RALEF-2D: excerpts from the main report.

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Contents

1. THERMAL CONDUCTION

1. Spitzer thermal conductivity

The Spitzer coefficient of electron thermal conductivity in a plasma of hydrogen-like ions $+ez_i$, as originally presented by L. Spitzer [1], is written as

$$\kappa_e = 20\left(\frac{2}{\pi}\right)^{3/2} \frac{T_e^{5/2}}{m_e^{1/2}e^4 z_i\,\ln\Lambda}\,\epsilon\delta_T. \tag{1.1}$$

An analytic expression for the factor $\epsilon\delta_T$ was first calculated by V.S. Imshennik [2] using the Chapman-Enskog method in the two-polynomial approximation. Later, it was independently obtained by M. Lampe [3, 4], and still later recalculated by N. Bobrova and P. Sasorov [5]. The final result, as first given by Brysk *et al.* [6], reads

$$\epsilon\delta_T = \frac{15\pi}{256}\,\frac{45\zeta+433\zeta^2}{9+151\zeta+217\zeta^2}, \quad \zeta = \frac{1}{4\sqrt{2}}\frac{z_i\,\ln\Lambda_{ei}}{\ln\Lambda_{ee}}. \tag{1.2}$$

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If we assume $\ln \Lambda_{ei} = \ln \Lambda_{ee} = \ln \Lambda$, we get

$$\begin{aligned} \kappa_e &= \frac{3 \cdot 5^3}{2^7 \sqrt{\pi}} \frac{1 + \frac{433}{180\sqrt{2}} z_i}{1 + \frac{151}{36\sqrt{2}} z_i + \frac{217}{288} z_i^2} \frac{T_e^{5/2}}{m_e^{1/2} e^4 \ln \Lambda} \\ &\overset{\mathrm{R}}{=} 1.028908 \times 10^{51} \frac{[t]^3 [T]^{7/2}}{[m][l]} \frac{1 + \frac{433}{180\sqrt{2}} z_i}{1 + \frac{151}{36\sqrt{2}} z_i + \frac{217}{288} z_i^2} \frac{T_e^{5/2}}{\ln \Lambda}. \end{aligned} \tag{1.3}$$

The result of Imshennik differs slightly from Eq. (1.3) in that the factor by z_i^2 is 212/288 instead of the correct value of 217/288.

To evaluate the Coulomb logarithm $\ln \Lambda$, we follow the original work of Spitzer [1]. We start with the formula

$$\Lambda = \frac{v/\omega_{pe}}{\left[\left(z_i e^2 / 1.123 m_e v^2 \right)^2 + \left(\hbar / 2 m_e v \right)^2 \right]^{1/2}}, \tag{1.4}$$

which combines the Bohr-Kramers classical formula with the Bethe-Lindhard-Larkin quantum formula for a fast ion $z_i e$ moving with velocity v past motionless plasma electrons. Next, we change to the ion rest frame, replace the adiabatic impact parameter v/ω_{pe} (ω_{pe} is the plasma frequency) in the numerator with the Debye length λ_D given by

$$\lambda_D^{-2} = \frac{4\pi n_e e^2}{T_e} + \frac{4\pi n_i z_i^2 e^2}{T_i} = 4\pi n_e e^2 \left(\frac{1}{T_e} + \frac{z_i}{T_i} \right), \tag{1.5}$$

and replace $m_e v^2$ in the denominator with $3T_e$. In result we obtain

$$\begin{aligned} \Lambda &= 3.369 \frac{T_e}{e^2} \left[\lambda_D^{-2} \left(z_i^2 + 0.945847 \, \hbar^2 T_e / m_e e^4 \right) \right]^{-1/2} \\ &\overset{\mathrm{R}}{=} 1.105164 \times 10^{16} \frac{([l][T])^{3/2}}{[m]^{1/2}} T_e \left[\left(\frac{\rho z_i}{A} \right) \left(\frac{1}{T_e} + \frac{z_i}{T_i} \right) \left(z_i^2 + 2.1695 \times 10^{10} \, [T] \, T_e \right) \right]^{-1/2} \end{aligned} \tag{1.6}$$

When a flux limit is applied to the electron thermal conduction, the limiting heat flux h_l [erg cm^{-2} s^{-1}] is written in the form

$$h_l = f_{inh} \, n_e T_e \left(\frac{T_e}{m_e} \right)^{1/2} \overset{\mathrm{R}}{=} 1.99529 \times 10^{37} \frac{[t]^3 [T]^{3/2}}{[l]^3} f_{inh} \left(\frac{\rho z_i}{A} \right) T_e^{3/2}, \tag{1.7}$$

where $f_{inh} \simeq 0.03$–1 is a dimensionless inhibition factor.

In the RALEF package the Spitzer conductivity is implemented as model # 3 in subroutine `TCCOEF`, file `'f06_eos.f'`.

2. A semi-empirical model for thermal conductivity of mono-atomic metals and plasmas in the mean ion approximation

Starting from the Spitzer formula, a semi-empirical formula can be proposed for thermal conductivity of simple (mono-atomic) metals and plasmas, which is based on the mean ion approximation and matches the high-temperature Spitzer plasma limit with the measured conductivity at normal conditions by choosing an appropriate value of a single fitting parameter. It is implemented as model # 5 in subroutine `TCCOEF` (file '`f06_eos.f`') of the RALEF package.

The mono-atomic plasma of an element (A, Z) with

$$n = \frac{\rho}{m_u A} \tag{1.8}$$

nuclei per unit volume is characterized by a mean ionization degree

$$z_{ion} = \frac{n_e}{n}, \quad 0 \le z_{ion} \le Z, \tag{1.9}$$

(provided by the EOS model), and is assumed to consist of identical point-like ions with a generally fractional charge

$$\tilde{z}_i = \max(1, z_{ion}) = \begin{cases} z_{ion}, & z_{ion} \ge 1, \\ 1, & z_{ion} < 1, \end{cases} \tag{1.10}$$

and with the number density

$$n_i = \begin{cases} n, & z_{ion} \ge 1, \\ n z_{ion} = n_e, & z_{ion} < 1. \end{cases} \tag{1.11}$$

Then, the coefficient of thermal conductivity due to free electrons in such a plasma can be evaluated by generalizing the Spitzer formula (1.3) to include (approximately) the effects of electron degeneracy and of the electron scattering by neutral atoms.

For the Coulomb logarithm $\ln\Lambda$, the degeneracy effects are accounted for by simply replacing T_e in Eq. (1.6) with

$$T_F = \sqrt{T_e^2 + \left(\frac{2}{3}E_F\right)^2}, \tag{1.12}$$

where

$$E_F = \frac{\hbar^2}{2m_e}\left(3\pi^2 n_e\right)^{2/3} \overset{\text{R}}{=} 4.16628 \times 10^{-11} \frac{[m]^{2/3}}{[l]^2[T]} \left(\frac{\rho z_{ion}}{A}\right)^{2/3} \tag{1.13}$$

is the Fermi energy, because in the degenerate case it is the electrons at the Fermi surface (in the narrow energy range $\Delta(\frac{1}{2}m_e v_e^2) \simeq T_e \ll E_F$) which determine the plasma transport properties, i.e. the electrons with the mean energy equal to E_F instead of $\frac{3}{2}T_e$ [6]. At the same time, the factor $T_e^{5/2}$ in the numerator of Eq. (1.3) must be split into $T_e \widetilde{T}_F^{3/2}$, where

$$\widetilde{T}_F = \sqrt{T_e^2 + (\beta_{ec} E_F)^2}, \quad \beta_{ec} = 0.34. \tag{1.14}$$

The numerical coefficient β_{ec} in Eq. (1.14) is adjusted such as to yield the correct value of κ_e in the limit of $T_e \ll E_F$ for the degenerate Lorentzian plasma [3, 6].

Inclusion of the electron-atom scattering is done by recalling that generally

$$\kappa_e = const \cdot \frac{n_e T_e}{m_e \nu_e} = const \cdot \frac{n z_{ion} T_e}{m_e \nu_e}, \tag{1.15}$$

where the electron collision frequency ν_e is the sum of the ei-collision and ea-collision terms

$$\nu_e = \nu_{ei} + \nu_{ea} = const \cdot \frac{e^4}{m_e^{1/2} \widetilde{T}_F^{3/2}} n z_{ion} \tilde{z}_i L_{ei} + n\sigma_{ea} \left(\frac{T_F}{m_e}\right)^{1/2} \max(0, 1 - z_{ion}); \qquad (1.16)$$

when evaluating the coefficient K_{ea} in Eq. (1.19), a fixed value of the electron-atom scattering cross-section $\sigma_{ea} = 10^{-15}$ cm^{-2} is used.

Finally, we obtain the following expression for the electron conduction coefficient

$$\begin{aligned} \kappa_e &= \frac{3 \cdot 5^3}{2^7 \sqrt{\pi}} \frac{\tilde{z}_i + \frac{433}{180\sqrt{2}} \tilde{z}_i^2}{1 + \frac{151}{36\sqrt{2}} \tilde{z}_i + \frac{217}{288} \tilde{z}_i^2} \frac{z_{ion} T_e \widetilde{T}_F^{3/2}}{m_e^{1/2} e^4 \left[z_{ion} \tilde{z}_i L_{ei} + K_{ea} \widetilde{T}_F^{3/2} T_F^{1/2} \max(0, 1 - z_{ion}) \right]} \\ &\overset{\mathrm{R}}{=} K_{ec} \frac{\tilde{z}_i + \frac{433}{180\sqrt{2}} \tilde{z}_i^2}{1 + \frac{151}{36\sqrt{2}} \tilde{z}_i + \frac{217}{288} \tilde{z}_i^2} \frac{z_{ion} T_e \widetilde{T}_F^{3/2}}{z_{ion} \tilde{z}_i L_{ei} + K_{ea} \widetilde{T}_F^{3/2} T_F^{1/2} \max(0, 1 - z_{ion})}, \end{aligned} \qquad (1.17)$$

where

$$K_{ec} \overset{\mathrm{R}}{=} 1.028908 \times 10^{51} \frac{[t]^3 [T]^{7/2}}{[m][l]} \overset{\mathrm{D}}{=} 1.693806, \qquad (1.18)$$

$$K_{ea} \overset{\mathrm{R}}{=} 5.621421 \times 10^{21} [T]^2 \overset{\mathrm{D}}{=} 1.4430 \times 10^4. \qquad (1.19)$$

For the Coulomb logarithm $\ln \Lambda_{ei}$ of ei collisions we use an interpolation expression

$$L_{ei} = \ln \left[1 + \frac{\Lambda_{ei}(g_{ec} + \Lambda_{ei})}{1 + \Lambda_{ei} + (6.5 g_{ec} \Lambda_{ei})^{-1}} \right], \qquad (1.20)$$

which has one free parameter g_{ec} for the limit of $\Lambda_{ei} \ll 1$ [7, 8], and becomes $\ln \Lambda_{ei}$ in the limit of $\Lambda_{ei} \gg 1$. The formula for Λ_{ei} is obtained from Eq. (1.6) by replacing T_e with T_F, and z_i with either z_{ion} (where appropriate) or $\tilde{z}_i$:

$$\begin{aligned} \Lambda_{ei} &= 3.369 \frac{T_F}{e^2} \left[\lambda_D^{-2} \left(\tilde{z}_i^2 + 0.945847\, \hbar^2 T_F / m_e e^4 \right) \right]^{-1/2} \\ &\overset{\mathrm{R}}{=} 1.105164 \times 10^{16} \frac{([l][T])^{3/2}}{[m]^{1/2}} T_F \left[\left(\frac{\rho z_{ion}}{A} \right) \left(\frac{1}{T_F} + \frac{\tilde{z}_i}{T_i} \right) \left(\tilde{z}_i^2 + 2.1695 \times 10^{10} \, [T] \, T_F \right) \right]^{-1/2} \end{aligned} \qquad (1.21)$$

When a flux limit is applied to the electron thermal conduction, the limiting heat flux h_l [erg cm^{-2} s^{-1}] is written in the form

$$h_l = f_{inh} \, n_e T_e \left(\frac{T_F}{m_e} \right)^{1/2} \overset{\mathrm{R}}{=} 1.99529 \times 10^{37} \frac{[t]^3 [T]^{3/2}}{[l]^3} f_{inh} \left(\frac{\rho z_{ion}}{A} \right) T_e \sqrt{T_F}, \qquad (1.22)$$

where $f_{inh} \simeq 0.03$–1 is a dimensionless inhibition factor.

The value of the only fitting parameter g_{ec} in Eq. (1.20) should normally be adjusted such as to reproduce the empirical value of thermal conductivity κ_0 in a liquid metal near normal conditions at $\rho = \rho_0$, $T = T_0 \simeq 300$–1000 K. It is assumed that the "ionization degree" (equal to the number of conduction electrons per atom and taken from the employed equation of state) z_0 under this conditions takes on a value $z_0 \gtrsim 1$. Since near normal conditions one typically has $T_e = T_i = T \ll T_F \ll z_0^2 (m_e e^4 / \hbar^2)$, the above formulae yield (near $z_0 \approx 3.5$)

$$\kappa_{0[W/K\ m]} \approx 200 \, \frac{z_0^2}{g_{ec}^2} \left(\frac{\rho_0 z_0}{A} \right)^{2/3}, \qquad (1.23)$$

where $\kappa_{0[W/K\ m]}$ is the tabular value of the conduction coefficient measured in W/K·m, and ρ_0 is measured in g/cm^3. The conversion formula from the conventional unit [W m^{-1} K^{-1}] for κ to the DEIRA system of units is

$$\kappa_{[DEIRA]} = 1.1604 \times 10^{-8}\, \kappa_{[W/K\ m]}. \tag{1.24}$$

Typically, the values of parameter g_{ec} fall in the range $g_{ec} \simeq 1$–10. For example, for liquid tin ($Z = 50$, $A = 118.7$, $\rho_0 \approx 6.85$ g/cm^3 at $T = 700$ K) the experimental value $\kappa_0 \approx 41$ W/K·m is obtained with $g_{ec} = 4.5$, when the value $z_0 = 3.5$ from the FEOS model is used.

2. LASER ABSORPTION COEFFICIENT

1. General inverse-bremsstrahlung absorption coefficient of radiation by Maxwellian plasma

As an example of opacity calculation, we consider the absorption coefficient for inverse bremsstrahlung [9]

$$k_{\nu,ff}(\nu, T_e) = \frac{32\pi^3}{3\sqrt{6\pi}}\,\alpha a_0^5 \left(\frac{e^2/a_0}{T_e}\right)^{1/2} \left(\frac{e^2/a_0}{h\nu}\right)^3 \left(1 - e^{-h\nu/T}\right) n_e n_i z_i^2\, \bar{g}, \tag{2.25}$$

in a plasma of hydrogen-like ions with an electric charge $+ez_i$; here e is the elementary charge, $\alpha = e^2/\hbar c = 1/137.036$ is the fine structure constant, $a_0 = \hbar^2/m_e e^2 = 0.52918 \times 10^{-8}$ cm is the Bohr radius, $n_e = z_i n_i$ is the number of free electrons per unit volume, $n_i = \rho/A m_u$ is the number of ions per unit volume, $\bar{g}$ is the mean value of the Gaunt factor. This absorption coefficient is already corrected for stimulated emission.

From Eq. (2.25) one readily calculates the absorption coefficient, averaged over a spectral interval $\nu \in [\nu_k, \nu_{k+1}]$ with the Planckian weight function $B_\nu \propto \nu^3 \left(e^{h\nu/T} - 1\right)^{-1}$ (the Planckian mean)

$$\begin{aligned} k_{ff,[k]} &\overset{\mathrm{R}}{=} K_{ff} \left(\frac{\rho}{A}\right)^2 \frac{z_i^3 \langle g\rangle_{[k]}}{T^{1/2}}\, \frac{\int\limits_{\nu_k}^{\nu_{k+1}} e^{-\nu/T}\, d\nu}{\int\limits_{\nu_k}^{\nu_{k+1}} \nu^3 \left(e^{\nu/T} - 1\right)^{-1} d\nu} \\ &\overset{\mathrm{R}}{=} K_{ff} \left(\frac{\rho}{A}\right)^2 \frac{z_i^3 \langle g\rangle_{[k]}}{T^{7/2}}\, \frac{e^{-x_k} - e^{-x_{k+1}}}{\Phi(x_k) - \Phi(x_{k+1})}, \end{aligned} \tag{2.26}$$

where

$$K_{ff} \overset{\mathrm{R}}{=} 4.577350 \times 10^{-31} \frac{[m]^2}{[l]^5 [T]^{7/2}} \overset{\mathrm{D}}{=} 0.2780532, \quad x_k = \frac{\nu_k}{T}, \tag{2.27}$$

$\langle g\rangle_{[k]}$ is the group-mean value of the Gaunt factor, and

$$\Phi(x) = \int\limits_x^\infty \frac{t^3}{e^t - 1}\, dt. \tag{2.28}$$

For fast numerical evaluation one can use an approximation

$$\frac{t}{1-e^{-t}} \approx t + e^{-t/2}, \tag{2.29}$$

which yields

$$\Phi(x) \approx (x^3 + 3x^2 + 6x + 6)e^{-x} + \frac{2}{3}\left(x^2 + \frac{4}{3}x + \frac{8}{9}\right)e^{-3x/2}. \tag{2.30}$$

The relative error of the approximation (2.29) does not exceed 2.4%.

The full (i.e. over the entire spectrum $0 \leq \nu \leq \infty$) Planckian and Rosseland mean absorption coefficients are given by

$$k_{ff,P} \overset{\text{R}}{=} \frac{15}{\pi^4} K_{ff} \left(\frac{\rho}{A}\right)^2 \frac{z_i^3 \langle g \rangle}{T^{7/2}} \overset{\text{R}}{=} 0.15398973382\, K_{ff} \left(\frac{\rho}{A}\right)^2 \frac{z_i^3 \langle g \rangle}{T^{7/2}}, \tag{2.31}$$

$$\begin{aligned} k_{ff,R} &\overset{\text{R}}{=} \frac{4\pi^4}{15} \left(\int_0^\infty \frac{x^7 e^{-x}}{(1-e^{-x})^3}\, dx \right)^{-1} K_{ff} \left(\frac{\rho}{A}\right)^2 \frac{z_i^3 \bar{\bar{g}}}{T^{7/2}} \\ &\overset{\text{R}}{=} 0.00508855177\, K_{ff} \left(\frac{\rho}{A}\right)^2 \frac{z_i^3 \bar{\bar{g}}}{T^{7/2}}; \end{aligned} \tag{2.32}$$

here $\langle g \rangle$ and $\bar{\bar{g}}$ are the corresponding mean values of the Gaunt factor.

To describe the absorption of laser light with $h\nu \lesssim 3$ eV in hot plasmas, usually the monochromatic formula (2.25) in the limit of $h\nu \ll T$ is used. In our context it can be written as

$$k_{\nu,ff} \overset{\text{R}}{=} K_{ff} \left(\frac{\rho}{A}\right)^2 \frac{z_i^3\, \bar{g}}{T^{3/2}\, \nu^2}. \tag{2.33}$$

The Maxwellian-mean Gaunt factor $\bar{g}$ is most readily calculated in the Born approximation

$$\bar{g} = \frac{\sqrt{3}}{\pi} \exp\left(\frac{h\nu}{2T}\right) K_0\left(\frac{h\nu}{2T}\right), \tag{2.34}$$

where $K_0(x)$ is the Macdonald function of order zero. In the limit of $h\nu \ll T$ we can use the corresponding asymptotic expansion of $K_0(x)$, which leads us to a simple approximation

$$\bar{g} \overset{\text{R}}{=} \max\left\{1;\ \frac{\sqrt{3}}{\pi} \ln\left(2.24584\, \frac{T}{\nu}\right)\right\}. \tag{2.35}$$

2. Complex dielectric permittivity within the Drude model

a. The Drude model. Within the Drude model, the complex dielectric permittivity of a plasma

$$\varepsilon(\omega) = \varepsilon' + i\varepsilon'' \tag{2.36}$$

is expressed in terms of the effective collision frequency ν_e of the free electrons as

$$\varepsilon' = 1 - \frac{\bar{\omega}_{pe}^2}{1+\bar{\nu}_e^2}, \quad \varepsilon'' = \frac{\bar{\omega}_{pe}^2 \bar{\nu}_e}{1+\bar{\nu}_e^2}, \tag{2.37}$$

where $\varepsilon' > -\infty$ and $\varepsilon'' > 0$ are real numbers, ω is the angular frequency of the laser light, $i = \sqrt{-1}$,

$$\bar{\nu}_e = \frac{\nu_e}{\omega}, \quad \bar{\omega}_{pe}^2 = \frac{\omega_{pe}^2}{\omega^2} = \frac{n_e}{n_{e,cr}}, \quad \omega_{pe}^2 = \frac{4\pi n_e e^2}{m_e}, \quad n_{e,cr} = \frac{m_e \omega^2}{4\pi e^2}, \tag{2.38}$$

n_e is the number density of free electrons. Possible contribution of the bound electrons to the dielectric permittivity is neglected. Generally, collisions of free electrons with charged ions and neutral atoms are taken into account. Electron-electron collisions are ignored because they do not contribute to the dipole emission (absorption) of electromagnetic waves. In the RALEF package (the subroutines `LASKNU` and `LASEPSN` in file `'f06_eos.f'`) the present model is implemented as model # 5.

The laser absorption coefficient k_{las} is expressed in terms of the imaginary part κ'' of the complex refraction index $\kappa = \sqrt{\varepsilon}$ as

$$k_{las} = 2\,\frac{\omega}{c}\,\kappa'' \overset{\text{R}}{=} 6.3260586 \times 10^{16}\,[l][T]\,\nu_{las}\,\kappa'', \tag{2.39}$$

where $\overset{\text{R}}{=}$ means that the RALEF-code units of measurement are used, and

$$\kappa' = \left(\frac{\sqrt{\varepsilon''^2 + \varepsilon'^2} + \varepsilon'}{2}\right)^{1/2} = \frac{\varepsilon''}{2\kappa''}, \tag{2.40}$$

$$\kappa'' = \left(\frac{\sqrt{\varepsilon''^2 + \varepsilon'^2} - \varepsilon'}{2}\right)^{1/2} = \frac{\varepsilon''}{2\kappa'}. \tag{2.41}$$

For numerical evaluation, the following implementation of the above formulae is used

Case $\varepsilon' \geq 0$:

$$\kappa' = \begin{cases} \sqrt{\varepsilon'}, & 10^4\varepsilon'' \leq \varepsilon', \\ \left[0.5\left(\sqrt{\varepsilon''^2 + \varepsilon'^2} + \varepsilon'\right)\right]^{1/2}, & 0 \leq \varepsilon' < 10^4\varepsilon'', \end{cases} \tag{2.42}$$
$$\kappa'' = 0.5\varepsilon''/\kappa';$$

Case $\varepsilon' < 0$:

$$\kappa'' = \begin{cases} \sqrt{-\varepsilon'}, & \varepsilon' \leq -10^4\varepsilon'', \\ \left[0.5\left(\sqrt{\varepsilon''^2 + \varepsilon'^2} - \varepsilon'\right)\right]^{1/2}, & -10^4\varepsilon'' < \varepsilon' < 0, \end{cases} \tag{2.43}$$
$$\kappa' = 0.5\varepsilon''/\kappa''.$$

b. Maxwellian plasma of hydrogen-like ions. First consider the simplest case of a plasma, composed of identical ions with a discrete charge z_i ($1 \leq z_i \leq Z$) at a number density

$$n_i = n = \frac{\rho}{m_u A}, \tag{2.44}$$

and free electrons with a number density

$$n_e = n_i z_i = \frac{\rho z_i}{m_u A}. \tag{2.45}$$

Assuming ions to be infinitely heavy point-like charges, the electron collision frequency ν_e is given by a well-known [10, §44, 48] expression

$$\nu_e \equiv \nu_{ei} = \frac{4\sqrt{2\pi}}{3} \frac{e^4 n_i z_i^2}{m_e^{1/2} T_e^{3/2}} L_{ei}, \tag{2.46}$$

where the Coulomb logarithm L_{ei} takes a different form in the low-frequency (an overdense plasma with $\omega \ll \omega_{pe}$) [10, §44] and the high-frequency (an underdense plasma with $\omega \gg \omega_{pe}$) [10, §48] limits: in particular, in the case of applicability of the quantum theory of Coulomb scattering

$$L_{ei} = \begin{cases} \ln\left(\dfrac{aT_e}{\hbar\omega_{pe}}\right), & \omega \ll \omega_{pe}, \\ \ln\left(\dfrac{4}{\gamma}\dfrac{T_e}{\hbar\omega}\right), & \omega \gg \omega_{pe}, \end{cases} \tag{2.47}$$

where $\ln\gamma = 0.5772\ldots$ is Euler's constant, and a is an unknown numerical factor stemming from averaging the Landau collision integral. In the high-frequency limit the value of L_{ei} is obtained from the Gaunt factor in the first Born approximation.

To obtain a more general formula for the Coulomb logarithm $L_{ei} = \ln\Lambda_{ei}$ of ei collisions, matching smoothly the classical and the quantum limits, we proceed as follows. We start with the formula

$$\Lambda_{ei} = \frac{v/\omega_{pe}}{\left[(z_i e^2/1.123 m_e v^2)^2 + (\hbar/2m_e v)^2\right]^{1/2}}, \tag{2.48}$$

which combines the Bohr-Kramers classical formula with the Bethe-Lindhard-Larkin quantum formula for a fast ion $+ez_i$ moving with velocity v past motionless plasma electrons. Next, we change to the ion rest frame and, following the original work of Spitzer [1], replace the adiabatic impact parameter v/ω_{pe} in the numerator with the Debye length λ_D, given by

$$\lambda_D^{-2} = \frac{4\pi n_e e^2}{T_e} + \frac{4\pi n_i z_i^2 e^2}{T_i} = m_e\omega_{pe}^2\left(\frac{1}{T_e} + \frac{z_i}{T_i}\right), \tag{2.49}$$

and replace $m_e v^2$ in the denominator with $3T_e$. The resulting formula

$$\Lambda_{ei} = 2\sqrt{3}\,\frac{T_e}{\hbar}\left[\left(\omega_{pe}^2 + \omega_{pe}^2\frac{z_i T_e}{T_i}\right)\left(1 + 1.0574\frac{z_i^2 m_e e^4}{\hbar^2 T_e}\right)\right]^{-1/2}, \tag{2.50}$$

taken in the quantum limit of $z_i e^2/\hbar v \ll 1$, yields the low-frequency case $\omega \ll \omega_{pe}$ in Eq. (2.47) with $a = 2\sqrt{3}(1+z_i)^{-1/2}$.

To describe the high-frequency limit $\omega \gg \omega_{pe}$, we replace the first occurrence of ω_{pe}^2 in Eq. (2.50) with

$$\omega_{pe}^2 \to \omega_{pe}^2 + \beta_\omega^2\omega^2, \quad \beta_\omega = \frac{\sqrt{3}}{2}\gamma \approx 1.542454. \tag{2.51}$$

In result, the final formula for Λ_{ei} takes the form

$$\Lambda_{ei} = 2\sqrt{3}\,\frac{T_e}{\hbar\omega}\left\{\left[\beta_\omega^2 + \bar{\omega}_{pe}^2\left(1 + \frac{z_i T_e}{T_i}\right)\right]\left(1 + 1.0574\frac{z_i^2 m_e e^4}{\hbar^2 T_e}\right)\right\}^{-1/2}. \tag{2.52}$$

c. Monoatomic plasma of elements in the mean ion approximation. Consider a plasma consisting of atoms (nuclei) of one sort with the atomic mass A, atomic number Z, and the total number density of nuclei (neutral atoms plus ions) given by

$$n = \frac{\rho}{m_u A}. \tag{2.53}$$

We assume that this plasma has a mean ionization degree $0 \le z_{ion} \le Z$, defined as

$$n_e = n z_{ion}, \quad z_{ion} = n^{-1} \sum_{j=1}^{Z} j\, n_{ij}, \tag{2.54}$$

where n_{ij} is the number density of ions with charge $+ej$. It is assumed that the values of $z_{ion}(\rho, T)$ are provided by the EOS model.

Further on, within the approximation of mean ion we assume that for $z_{ion} \ge 1$ all the plasma ions are identical point-like charges with a fractional charge $+ez_{ion}$, and the hydrogenic formulae from the previous subsection apply with $z_i = z_{ion}$. For $z_{ion} < 1$ we assume the plasma to be a mixture of singly charged ions and neutral atoms with number densities

$$n_i = n_{i1} = n z_{ion} = n_e, \quad n_a = n_{i0} = n(1 - z_{ion}). \tag{2.55}$$

Accordingly, the electron collision frequency

$$\nu_e = \nu_{ei} + \nu_{ea} \tag{2.56}$$

becomes the sum of those with the singly charged ions, i.e. ν_{ei} from Eq. (2.46) with $n_i = n z_{ion}$, $z_i = 1$, and with the neutral atoms

$$\nu_{ea} = \sigma_{ea} n_a \left(\frac{T_e}{m_e}\right)^{1/2} = \sigma_{ea} n \left(\frac{T_e}{m_e}\right)^{1/2} \max(0, 1 - z_{ion}), \tag{2.57}$$

where for the cross-section of ea collisions we assume a fixed value of $\sigma_{ea} = 10^{-15}$ cm^{-2}. In the end, the two cases of $z_{ion} \ge 1$ and $z_{ion} < 1$ can be combined within a universal hydrogen-like formula for ν_{ei} from the previous subsection by setting

$$n_e = n z_{ion}, \quad n_i z_i^2 = n_e \tilde{z}_i, \quad \tilde{z}_i = \max(1, z_{ion}), \tag{2.58}$$

and replacing all the alone-standing z_i under the Coulomb logarithm with $\tilde{z}_i$.

The effects of the electron degeneracy are accounted separately in the pre-logarithmic term in Eq. (2.46) and under the Coulomb logarithm. In the Coulomb logarithm $\ln \Lambda_{ei}$, it is sufficient to simply replace T_e in Eq. (2.52) with

$$T_F = \sqrt{T_e^2 + \left(\frac{2}{3} E_F\right)^2}, \tag{2.59}$$

where

$$E_F = \frac{\hbar^2}{2m_e} \left(3\pi^2 n_e\right)^{2/3} \overset{\text{R}}{=} 4.16628 \times 10^{-11} \frac{[m]^{2/3}}{[l]^2 [T]} \left(\frac{\rho z_{ion}}{A}\right)^{2/3} \tag{2.60}$$

is the Fermi energy, because in the degenerate case it is the electrons at the Fermi surface (in the narrow energy range $\Delta(\frac{1}{2} m_e v_e^2) \simeq T_e \ll E_F$) which determine the plasma transport properties, i.e. the electrons with the mean energy equal to E_F instead of $\frac{3}{2}T_e$ [6]. Also, T_e in Eq. (2.57) for the *ea* collision frequency should be replaced by T_F. At the same time, the term $T_e^{3/2}$ in the denominator of Eq. (2.46) must be replaced by $\widetilde{T}_F^{3/2}$, where

$$\widetilde{T}_F = \sqrt{T_e^2 + (\beta_{\nu ei} E_F)^2}, \quad \beta_{\nu ei} = \left(\frac{4}{3\sqrt{\pi}}\right)^{2/3} \approx 0.82713. \tag{2.61}$$

The numerical coefficient $\beta_{\nu ei}$ in Eq. (2.61) is adjusted such as to recover the correct value of ν_{ei} in the degenerate limit of $T_e \ll E_F$ for the Lorentzian plasma [6, 11]. Finally, we arrive at the following formulae

$$\bar{\omega}_{pe}^2 = \left(\frac{\rho z_{ion}}{A}\right)\left(\frac{\rho z_{ion}}{A}\right)_{cr}^{-1}, \quad \left(\frac{\rho z_{ion}}{A}\right)_{cr} \overset{\mathrm{R}}{=} 4.691522 \times 10^{20} \frac{[T]^2[l]^3}{[m]} \nu_{las}^2, \tag{2.62}$$

$$\bar{\nu}_e \overset{\mathrm{R}}{=} 3.743144 \times 10^{-27} \frac{[m]}{[l]^3[T]^{5/2}} \frac{\rho}{A\widetilde{T}_F^{3/2}\nu_{las}} \left[z_{ion}\tilde{z}_i L_{ei} + K_{ea}\widetilde{T}_F^{3/2} T_F^{1/2} \max(0, 1 - z_{ion})\right], \tag{2.63}$$

$$K_{ea} \overset{\mathrm{R}}{=} 5.621421 \times 10^{21} [T]^2, \tag{2.64}$$

$$\Lambda_{ei} \overset{\mathrm{R}}{=} 2\sqrt{3}\left(\frac{T_F}{\nu_{las}}\right)\left\{\left[\beta_\omega^2 + \bar{\omega}_{pe}^2\left(1 + \frac{\tilde{z}_i T_F}{T_i}\right)\right]\left(1 + \frac{4.610 \times 10^{-11}}{[T]} \frac{\tilde{z}_i^2}{T_F}\right)\right\}^{-1/2}. \tag{2.65}$$

For the Coulomb logarithm $\ln \Lambda_{ei}$ of *ei* collisions we use an interpolation formula

$$L_{ei} = \ln\left[1 + \frac{\Lambda_{ei}(g_\epsilon + \Lambda_{ei})}{1 + \Lambda_{ei} + (6.5 g_\epsilon \Lambda_{ei})^{-1}}\right], \tag{2.66}$$

which has one free parameter g_ϵ for the limit of $\Lambda_{ei} \ll 1$ [7], and approaches $\ln \Lambda_{ei}$ in the limit of $\Lambda_{ei} \gg 1$ independently of the g_ϵ value. Formula (2.66) has a physically reasonable behavior $L_{ei} \propto \Lambda_{ei}^2$ in the limit of $\Lambda_{ei} \ll 1$, which corresponds to the scattering of free electrons on thermal ion oscillations in a crystal lattice of a metal [7]. The fit parameter g_ϵ is used to achieve agreement with the experimental reflection coefficient of a given metal near normal conditions for a given laser frequency ω; for Sn, a practically perfect agreement with the data from Ref. [12] at $T = 335$ °C, $\hbar\omega = 0.1$-1 eV is achieved with $g_\epsilon = 8.3$ ($f_a = 7.00\%$ versus 6.95% in [12] for $\hbar\omega = 0.1$ eV, and $f_a = 15.17\%$ versus 15.54% in [12] for $\hbar\omega = 1$ eV).

3. The inverse-bremsstrahlung collisional absorption of laser light

As a rule, the absorption of laser light with $\hbar\omega \equiv h\nu_{las} \lesssim 3$–10 eV in hot undercritical ($\omega \gg \omega_{pe}$) plasmas can be described by the monochromatic Kramers formula (2.25). In the subroutines `LASKNU` and `LASEPSN` (file `'f06_eos.f'`) of the RALEF package this model is implemented as model # 3. In the context of the mean ion approximation to the monoatomic plasma (see the previous subsection), the formula (2.25) can be rewritten as

$$k_{las,ff} \overset{\mathrm{R}}{=} K_{ff}\left(\frac{\rho^2 z_{ion}}{A^2}\right)\frac{1 - e^{-\nu_{las}/T_e}}{T_e^{1/2}\nu_{las}^3}\left[z_{ion}\tilde{z}_i\bar{g} + \frac{\sqrt{3}}{\pi}K_{ea}T_e^2 \max(0, 1 - z_{ion})\right]\Gamma_{las}, \tag{2.67}$$

where

$$K_{ff} \overset{\mathrm{R}}{=} 4.577350 \times 10^{-31} \frac{[m]^2}{[l]^5[T]^{7/2}}, \tag{2.68}$$

and z_{ion}, $\tilde{z}_i$, and K_{ea} are defined in the previous subsection. Equation (2.67) accounts for the collisions of free electrons with neutral atoms in the same simplified way as Eq. (2.63).

The ad hoc numerical factor Γ_{las} is introduced to imitate the enhanced laser absorption in the vicinity of the critical surface, where the free electron density $n_e \simeq n_{e,cr}$. In practice we use the approximation [13], [14, §5.1]

$$\Gamma_{las}(x) = \begin{cases} \left(1-\bar{\omega}_{pe}^2\right)^{-1/2}, & \bar{\omega}_{pe}^2 < 1-\Gamma_0^{-2}, \\ \Gamma_0, & \bar{\omega}_{pe}^2 \geq 1-\Gamma_0^{-2}, \end{cases} \tag{2.69}$$

where $\Gamma_0 \simeq 10$–100 is some reasonably large number. The value of Γ_0 is loaded as the input variable `ovcrkblas(iblas)` in subroutine `LASINPT`, file '`f10_taskinpt.f`'.

The Maxwellian-mean Gaunt factor $\bar{g}$ is most readily calculated in the Born approximation

$$\bar{g} = \frac{\sqrt{3}}{\pi} \exp\left(\frac{\hbar\omega}{2T_e}\right) K_0\left(\frac{\hbar\omega}{2T_e}\right), \tag{2.70}$$

where $K_0(z)$ is the Macdonald function of order zero. In the high-temperature limit $\hbar\omega \ll T$ one can use the corresponding asymptotic expansion of $K_0(z)$, which leads us to a simple approximation

$$\bar{g} \overset{\mathrm{R}}{=} \max\left\{1;\ \frac{\sqrt{3}}{\pi} \ln\left(2.24584\,\frac{T_e}{\nu_{las}}\right)\right\}. \tag{2.71}$$

3. USER-DEFINED UNITS IN THE RALEF CODE

1. The general RALEF units

Equations of hydrodynamics (??)–(??) have no physical constants and preserve their form for any self-consistent system of units based on three fundamental units of measurement

$$\begin{aligned} \text{length unit} &= [l] \text{ centimeters}, \\ \text{time unit} &= [t] \text{ seconds}, \\ \text{mass unit} &= [m] \text{ grams}, \end{aligned} \tag{3.72}$$

which can be chosen arbitrarily. In Eq. (3.72) and everywhere below we assume that numerically the freely chosen fundamental units $[m]$, $[l]$ and $[t]$ are always expressed in the CGS (centimeter-gram-second) units. The system of units is called self-consistent when the units for all the other physical quantities in Eqs. (??)–(??) are derived from the base units (3.72) as listed in Table 3.1.

Once everybody agrees to use only self-consistent systems of units, the units of measurement do not have to be specified in a pure hydrodynamic code. However, the situation changes after the heat conduction term and the radiation transfer equation are added to the basic equations. First of all, the temperature T emerges as one of the principal variables, and dimensional physical constants appear in expressions for the heat conduction coefficient $\kappa = \kappa(\rho, T)$, the Planckian intensity $B_\nu = B_\nu(\nu, T)$, and the radiation absorption coefficient $k_\nu = k_\nu(\nu, \rho, T)$. Evidently, the numerical values of dimensional constants will be different in different systems of user-defined units. The speed of light, for example, will be equal to

$$c \overset{\mathrm{R}}{=} 2.99792458 \times 10^{10}\,\frac{[t]}{[l]}. \tag{3.73}$$

TABLE 3.1: Derived units of measurement in hydrodynamic equations.

quantity	notation	unit	dimensions in CGS
velocity	$\vec{u}$	$[l][t]^{-1}$	cm s^{-1}
density	ρ	$[m][l]^{-3}$	g cm^{-3}
pressure	p	$[m][l]^{-1}[t]^{-2}$	erg cm^{-3}
mass-specific energy	E, e	$[l]^2[t]^{-2}$	erg g^{-1}
energy flux	$\vec{h}, \vec{h}_r$	$[m][t]^{-3}$	erg cm^{-2} s^{-1}
volume-specific power	Q_T, Q_r, Q_{dep}	$[m][l]^{-1}[t]^{-3}$	erg cm^{-3} s^{-1}

For practical (and/or historical) reasons, it is convenient to introduce the fourth independent base unit of measurement for temperature

$$\text{temperature unit} \;=\; [T] \text{ ergs.} \tag{3.74}$$

In the RALEF-2D code, the four base units (3.72) and (3.74) are treated as user-defined free parameters, and all dimensional formulae are cast in a form, which yields correct values for any choice of the four base units. With regard to the radiation frequency ν, we assume that the photon energy $h\nu$ is measured in the same energy units $[T]$ as the temperature T; in other words, if, for example, temperature is measured in eV (i.e. $[T] = 1.60217733 \times 10^{-12}$), then the radiation frequency ν (i.e. the photon energy $h\nu$) is also measured in eV.

Below the system of units, built upon the four base units (3.72) and (3.74) with a freely chosen set of the $[m]$, $[l]$, $[t]$, and $[T]$ values, is called the *RALEF-code units*. Table 3.2 lists several secondary (derived) units expressed in terms of the four base ones for selected physical quantities, related to thermal properties, heat and radiation transport. Generally, certain equations and formulae — like the basic radiation hydrodynamics equations (**??**)–(**??**) — remain invariant when the base units $[m]$, $[l]$, $[t]$, and $[T]$ are changed, but others do not. Below all the unit-dependent formulae that are cast in the form suitable for incorporation into the RALEF-code have the equality sign marked as $\stackrel{\text{R}}{=}$ [like in Eq. (3.73)].

TABLE 3.2: Derived units of measurement in heat and radiation transport.

quantity	notation	unit	dimensions in CGS
radiation frequency	ν	$h^{-1}[T]$	Hz
radiation intensity	I_ν, B_ν	$h[m][t]^{-3}[T]^{-1}$	erg cm^{-2} s^{-1} $ster^{-1}$ Hz^{-1}
group intensity	$B, \int I_\nu \, d\nu$	$[m][t]^{-3}$	erg cm^{-2} s^{-1} $ster^{-1}$
thermal conduction coefficient	κ	$[m][l][t]^{-3}[T]^{-1}$	cm^{-1} s^{-1}
mass-specific heat capacity	c_V	$[l]^2[t]^{-2}[T]^{-1}$	g^{-1}

The Planckian spectral and frequency-integrated radiation intensities in the RALEF-code units are given by

$$B_\nu \stackrel{\text{R}}{=} K_{Pl} \frac{\nu^3}{e^{\nu/T} - 1}, \quad B \stackrel{\text{R}}{=} K_{Pl} \int\limits_0^\infty \frac{\nu^3 \, d\nu}{e^{\nu/T} - 1}, \tag{3.75}$$

where

$$K_{Pl} \stackrel{\mathrm{R}}{=} \frac{2}{h^3c^2}\frac{[t]^3[T]^4}{[m]} \stackrel{\mathrm{R}}{=} 7.64926047 \times 10^{57}\, \frac{[t]^3[T]^4}{[m]}. \tag{3.76}$$

Expressions (**??**) for $B(T)$, (**??**) for the black-body flux $h_{r,Pl}$, and (**??**) for the black-body radiation energy density $\mathcal{E}_{r,Pl}$ preserve their form, provided that the Stefan-Boltzmann and the Stefan constants are given by

$$\sigma_{SB} \stackrel{\mathrm{R}}{=} \frac{\pi^5}{15}K_{Pl} \stackrel{\mathrm{R}}{=} 1.56054952 \times 10^{59}\, \frac{[t]^3[T]^4}{[m]}, \tag{3.77}$$

$$a_{St} \stackrel{\mathrm{R}}{=} 2.08217315 \times 10^{49}\, \frac{[l][t]^2[T]^4}{[m]}. \tag{3.78}$$

2. The DEIRA units.

In our notation, the parent CGS system of units corresponds to $[m] = [l] = [t] = 1$. These units are, however, rather inconvenient for most applications in the physics of high energy density states, where, for example, typical plasma velocities are often 6–8 orders of magnitude higher than 1 cm/s. A more suitable unit system would, say, be the one with $[m] = 10^{-3}$, $[l] = 0.1$, $[t] = 10^{-8}$, i.e. a system of units based on mass, length, and time units of 1 mg, 1 mm, and 10 ns. With this choice the mass density is measured in g/cm^3, the velocity in 10^7 cm/s, the pressure in 10^{14} erg/cm^3 = 100 Mbar, the mass-specific energy in 10^{14} erg/g = 10 MJ/g; the external heating rate Q_{dep} should be given in 10^{22} erg cm^{-3} s^{-1} = 1 PW/cm^3. Having added a temperature unit of $[T] = 1.60217733 \times 10^{-9} = 1$ keV, below we refer to these sytem as the DEIRA units — after the 1D DEIRA code, developed for simulation of heavy-ion driven ICF targets [8]. The unit-dependent formulae in the DEIRA units are marked by the equality sign $\stackrel{\mathrm{D}}{=}$. In the DEIRA units the values of the above mentioned constants become

$$K_{Pl} \stackrel{\mathrm{D}}{=} 50.403626, \quad a_{St} \stackrel{\mathrm{D}}{=} 1.372016, \quad \sigma_{SB} \stackrel{\mathrm{D}}{=} 1028.3001. \tag{3.79}$$

The DEIRA unit for the coefficient of thermal conduction κ is 10^{20} erg cm^{-1} s^{-1} keV^{-1}. The DEIRA units is the default option for the units of measurements in the present version of the RALEF-2D code.

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[1] L. Spitzer, *Physics of Fully Ionized Gases* (Interscience, New York, 1962), 2nd ed.

[2] V. S. Imshennik, Soviet Astronomy – AJ **5**, 495 (1962).

[3] M. Lampe, Phys. Rev. **170**, 306 (1968).

[4] M. Lampe, Phys. Rev. **174**, 276 (1968).

[5] N. A. Bobrova and P. V. Sasorov, Plasma Phys. Rep. **19**, 409 (1993).

[6] H. Brysk, P. M. Campbell, and P. Hammerling, Plasma Physics **17**, 473 (1975).

[7] M. M. Basko, T. Löwer, V. N. Kondrashov, A. Kendl, R. Sigel, and J. Meyer-ter Vehn, Physical Review E **56**, 1019 (1997).

[8] M. M. Basko, *DEIRA: A 1-D 3-T hydrodynamic code for simulating ICF targets driven by fast ion beams. Version 4* (2001; http://www.basko.net/mm/deira).

[9] Y. B. Zeldovich and Y. P. Raizer, *Physics of Shock-Waves and High-Temperature Hydrodynamic Phenomena* (Dover Pubn Inc, 2002), illustrated ed., ISBN 978-0486420028.

[10] L. Pitaevskii and E. Lifshitz, *Physical Kinetics*, v. 10 (Elsevier Science, 2012), ISBN 9780080570495, URL `http://books.google.ru/books?id=DTHxPDfV0fQC`.

[11] H. Brysk, Plasma Physics **16**, 927 (1974).

[12] G. Cisneros, J. S. Helman, and C. N. J. Wagner, Phys. Rev. B **25**, 4248 (1982).

[13] W. Kruer, *The physics of laser plasma interactions*, Frontiers in physics (Addison-Wesley, 1988), ISBN 9780201156720, URL `http://books.google.ru/books?id=csDvAAAAMAAJ`.

[14] S. Eliezer, *The Interaction of High-Power Lasers with Plasmas*, Series in Plasma Physics (CRC Press, 2002), ISBN 9781420033380, URL `https://books.google.ru/books?id=Hqzjvpu4kEUC`.