RADIATIVE TRANSFER IN A PLANE ATMOSPHERE WITH A STRONG MAGNETIC FIELD

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In a strong magnetic field the cross sections of elementary processes of interaction between radiation and matter depend on the angle between the field direction and the propagation of the electromagnetic waves. In this case the transfer equation can be solved by a technique which was developed for the investigation of radiative transfer in lines. The case of a plane-parallel atmosphere with a constant radiative flux and with an exponential distribution of primary sources is considered in detail. The direction of the emergent radiation and the distribution of plane radiant energy in the atmosphere are expressed in terms of Chandrasekhar's H-function. In the case of the limit of a strong field, $\nu/\nu_H \to 0$, the H-function is calculated numerically. Here $\nu_H = eH/2\pi m_e c$ is the electron gyrofrequency.

The generally accepted model of an x-ray pulsar in a binary system reduces to the following: in a rotating magnetized neutron star, matter falls, flowing with the normal of the component pair. The magnetic field directs the incident matter into the region of the magnetic poles which represent in this case the bright hot spots on the surface of the neutron star. The x-ray emission from these spots pulsates in the rotational frequency of the neutron star. A comparison with observed data shows that to explain the observed forms of the x-ray impulses, it is necessary to assume that the radiation of the hot spots is beamed.

If the magnetic field on the surface of a neutron star is sufficiently large, $H \gtrsim 10^{12}$-10^{13} G, and $\nu \ll \nu_H = eH/2\pi m_e c$, then the scattering cross section and radiation absorption become anisotropic [1]. In the present work we give a solution of the radiative transfer problem in a plane atmosphere with anisotropic scattering, where the angular dependence has the form (1)-(4). This solution is based the model of the pencil diagram of the direction of radiative accretion of x-ray pulsars discussed in detail in [2].

1. Cross Sections of Elementary Processes

Of the greatest practical interest are two fundamental processes which give a contribution to the opacity of an atmosphere: free—free absorption and Thomson scattering. In a magnetoactive plasma two types of electromagnetic waves are propagated independently of one another: "ordinary" and "extraordinary" — each with its own polarization. Within the limits of an extremely strong magnetic field, we are interested in the band of frequencies $\nu \ll \nu_H$, i.e., when $kT_e \ll h\nu_H$, and the generation of extraordinary waves can be neglected. The coefficient of free—free absorption and the differential cross section of scattering of ordinary waves within these limits take the form [1, 3]

$$k_{ff}(\nu, \theta) = k_{ff}^{(0)}(\nu) \sin^2\theta, \tag{1}$$

$$d\sigma_s(\theta \to \theta') = \frac{3}{8\pi} \sigma_T \sin^2\theta \sin^2\theta' d\Omega'. \tag{2}$$

Here and below it is assumed that the magnetic field is directed along the z axis which is normal to a plane semiinfinite atmosphere; the polar angles θ and θ' are measured with re-

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spect to the same axis; $k_{ff}^{(0)}(\nu)$ is the coefficient of free–free absorption in the absence of a magnetic field; $\sigma_T = 6.65 \cdot 10^{-25}$ cm^2 is the cross section of Thomson scattering.

The concept of a plane semiinfinite atmosphere with cross sections of the form (1)-(2) is not completely correct, since the optical depth of such an atmosphere along the direction $\theta = 0$ is $\tau_{ff} = \tau_S = 0$, while along any other direction $\theta = 0$ this thickness is $\tau_{ff} = \tau_s = \infty$. In order to dispose of this problem, and at the same time to estimate the influence of the small, but finite magnitude of the ratio ν/ν_H, we introduce a small parameter $\varepsilon = (\nu/\nu_H)^2$ and assume the following form for the angular dependence of the cross sections of the elementary processes:

$$k_{ff}(\nu, \theta) = k_{ff}^{(0)}[\sin^2\theta + \varepsilon], \tag{3}$$

$$d\sigma_s(\theta - \theta') = \frac{3}{8\pi}\left(1 + \frac{3}{2}\varepsilon\right)^{-1} \sigma_T (\sin^2\theta + \varepsilon)(\sin^2\theta' + \varepsilon)\, d\Omega. \tag{4}$$

Even if the expressions described do not represent the exact form of the decomposition of the corresponding cross sections up to the first order of smallness in $(\nu/\nu_H)^2$, nevertheless, they qualitatively correctly describe the behavior of these cross sections near small angles $\theta \ll 1$.

2. The Transfer Equation and Integral Equations of the Stationary State

The transfer equation in a plane semiinfinite atmosphere whose basic contributions to opacity are the free–free absorption and Thomson scattering with cross sections of the form (3)-(4), has the form

$$\mu \frac{\partial I(\nu, z, \mu)}{\partial z} = -(1 - \mu^2 + \varepsilon)[k_T(z) + k_{ff}(\nu, z)] I(\nu, z, \mu) +$$
$$+ (1 - \mu^2 + \varepsilon) k_T(z) \frac{3}{4}\left(1 + \frac{3}{2}\varepsilon\right)^{-1} \int_{-1}^{+1} (1 - \mu'^2 + \varepsilon) I(\nu, z, \mu')\, d\mu' +$$
$$+ (1 - \mu^2 + \varepsilon) k_{ff}(\nu, z) \frac{1}{2} B(\nu, T_e). \tag{5}$$

Here $\mu = \cos\theta$, $I(\nu, z, \mu)$ is the intensity of the radiation, $k_T(z) = \sigma_T N_e(z)$ and $k_{ff}(\nu, z)$ are the coefficients of opacity of the atmosphere in the absence of a magnetic field [we will drop the index "0" on $k_{ff}^{(o)}(\nu, z)$ in the sequel], and $B(\nu, T_e)$ is the intensity of equilibrium black-body radiation with temperature T_e. The multiplier 1/2 next to $B(\nu, T_e)$ takes into account the fact that we have neglected the generation of extraordinary waves and the field of equilibrium radiation contains only one polarization which consists of an ordinary wave.

We will solve Eq. (5) by using an approximation series of the atmosphere, i.e., we will assume that the coefficient of free–free absorption $k_{ff}(\nu, z) \equiv k_{ff}(z)$ and the constant ε are independent of the frequency. Introducing the optical depth

$$\tau = \int_z^\infty [k_T(z) + k_{ff}(z)]\, dz$$

as an independent variable, we rewrite (5) in the form

$$\frac{\mu}{1 - \mu^2 + \varepsilon} \frac{\partial I(\tau, \mu)}{\partial \tau} = I(\tau, \mu) - \frac{k_T}{k_T + k_{ff}} \frac{3}{4}\left(1 + \frac{3}{2}\varepsilon\right)^{-1} \int_{-1}^{+1} (1 - \mu'^2 + \varepsilon) I(\tau, \mu')\, d\mu - \frac{k_{ff}}{k_T + k_{ff}} \frac{\sigma T_e^4}{2\pi}. \tag{6}$$

Here $I(\tau, \mu) = \int_0^\infty I(\nu, \tau, \mu)\, d\nu$, and σ is the Stefan–Boltzmann constant.

In this paper an independent variable, characterizing a given layer of atmosphere, is used as the optical depth in which this layer can be situated in the absence of a magnetic field.

As is shown below, there exists an analogy between the considered problem and the problem of the transfer of radiation in a line under the assumption of a total redistribution according to the frequencies. In such an analogy the optical depth τ corresponds to the optical depth at the center of the line.

We let

$$L(\tau) = \frac{3}{4}\left(1+\frac{3}{2}\varepsilon\right)^{-1}\int_{-1}^{+1}(1-\mu^2+\varepsilon)\,I(\tau,\mu)\,d\mu \tag{7}$$

and introduce the function of the sources

$$S(\tau) = \frac{k_T}{k_T+k_{ff}}\,L(\tau) + \frac{k_{ff}}{k_T+k_{ff}}\,\frac{\sigma T_e^4}{2\pi}. \tag{8}$$

Then the transfer equation (6) can be written in the form

$$\frac{\mu}{1-\mu^2-\varepsilon}\,\frac{\partial I(\tau,\mu)}{\partial\tau} = I(\tau,\mu) - S(\tau). \tag{9}$$

Solving it for $I(\tau, \mu)$ under the usual boundary condition $I(o,\mu)|_{\mu<0} = 0$ and substituting the result into (7), we obtain

$$L(\tau) = \frac{1}{2}\int_0^{\infty} K_\varepsilon(|\tau-\xi|)\,S(\xi)\,d\xi, \tag{10}$$

where

$$K_\varepsilon(\tau) = \frac{3}{2}\left(1+\frac{3}{2}\varepsilon\right)^{-1}\int_0^1 \frac{(1-\mu^2+\varepsilon)^2}{\mu}\exp\left[-\frac{\tau}{\mu}(1-\mu^2+\varepsilon)\right]d\mu. \tag{11}$$

The kernel $K_\varepsilon(\tau)$ is normalized in the usual way:

$$\int_0^{\infty} K_\varepsilon(\tau)\,d\tau = 1.$$

Substituting (10) into (8), we obtain (in accordance with terminology used in [4]) an integral equation of the stationary state:

$$S(\tau) = \frac{k_T}{k_T+k_{ff}}\,\frac{1}{2}\int_0^{\infty} K_\varepsilon(|\tau-\xi|)\,S(\xi)\,d\xi + \frac{k_{ff}}{k_T+k_{ff}}\,\frac{\sigma T_e^4}{2\pi}. \tag{12}$$

If the temperature distribution $T_e(z)$ in the atmosphere is known, then the functions $k_T(\tau)$ and $k_{ff}(\tau)$ are known and the integral equation (12) is linear. Solving it, we find $S(\tau)$ and then with $S(\tau)$ known it is easy to find $I(\tau, \mu)$ from (9).

However, in astrophysics it is often the case that $T_e(z)$ is not known, but the source distribution of energy in the atmosphere is known. In this case (12) can be rewritten in the form

$$S(\tau) = \frac{1}{2}\int_0^{\infty} K_\varepsilon(|\tau-\xi|)\,S(\xi)\,d\xi - \frac{3}{8}\left(1+\frac{3}{2}\varepsilon\right)^{-1}\frac{dF(\tau)}{d\tau}. \tag{13}$$

Here $\pi F(\tau) = 2\pi \int_{-1}^{+1} \mu I(\tau, \mu)\, d\mu$ is the flux of radiant energy through a unit square normal to the z axis. Below we concentrate all of our attention on (13) which will be called the fundamental integral equation.

3. Solution of the Fundamental Integral Equation

In the present paper we consider the problem called the problem of transfer of radiation in a line: the photons far in the wings of the lines can freely escape from the deeper layers of the atmosphere. In a strong magnetic field photons freely leave from the deep layers for small angles θ if $\varepsilon < \theta^2$ (in particular, for $\varepsilon = 0$). This analogy permits us [which is different from similar equations which describe the transfer of radiation in lines and for monochromatic conservative scattering only by the type of kernel $K_\varepsilon(\tau)$] to solve (13) by using mathematical techniques which were developed in the works of V. V. Sobolev and K. Case; see [4] for a detailed bibliography. As will be seen from what follows, the mathematical relationship of the considered problem is the intermediate case between monochromatic scattering and transfer in a line. Therefore, there will be a comparison with these two extreme cases; the index "s" will denote all functions which describe monochromatic conservative scattering, and the indices "D" and "L" will denote functions which describe transfer in a line corresponding to a Doppler section (an absorption cross section proportional to $\exp[-(\nu/\nu_0)^2]$) and with a Lorentz section (an absorption cross section proportional to $[1 + (\nu/\nu_0)^2]^{-1}$).

The kernel of the integral equation $K_\varepsilon(\tau)$ is represented in standard form by the superposition of the exponentials

$$K_\varepsilon(\tau) = \int_0^\infty e^{-\tau x} G_\varepsilon(x) \frac{dx}{x}, \tag{14}$$

where $x = \mu/(1 - \mu^2 + \varepsilon)$, and

$$G_\varepsilon(x) = \begin{cases} \dfrac{3}{2}\left(1 + \dfrac{3}{2}\varepsilon\right)^{-1} \dfrac{\mu^3}{(1 + \mu^2 + \varepsilon)\, x^3}, & 0 \leqslant x \leqslant \varepsilon^{-1}; \\ 0, & x > \varepsilon^{-1}. \end{cases} \tag{15}$$

The asymptotic behavior of $G_{\varepsilon=0}(x) \equiv G_0(x)$ and $K_{\varepsilon=0}(\tau) \equiv K_0(\tau)$ for $x \to \infty$ and $\tau \to \infty$ in comparison with that studied previously (see, for example [4]) by similar asymptotics has the form

$$G_s(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1; \\ 0, & x > 1; \end{cases} \quad G_0(x) \sim \frac{3}{4x^3}; \quad G_D(x) \sim \frac{1}{2x^2\sqrt{\pi \ln x}};$$

$$G_L(x) \sim \frac{2}{3\pi x^{3/2}};$$

$$K_s(\tau) \sim \frac{e^{-\tau}}{\tau}; \quad K_0(\tau) \sim \frac{3}{2\tau^3}; \quad K_D(\tau) \sim \frac{1}{2\tau^2\sqrt{\pi \ln \tau}}; \quad K_L(\tau) \sim \frac{1}{3\sqrt{\pi}\,\tau^{3/2}}.$$

If $0 < \varepsilon \ll 1$, then for $1 \ll \tau \ll \varepsilon^{-1}$ the kernel $K_\varepsilon(\tau) \sim 3/2\tau^3$, and for $\tau \gg \varepsilon^{-1}$ it decreases exponentially, $K_\varepsilon(\tau) \sim (3/4)(\varepsilon^2/\tau)e^{-\tau\varepsilon}$. We note that the considered problem does not reduce completely to the problem of the transfer of radiation in a line with a definite section, since for this it it necessary that $G_\varepsilon(x) = \text{const}$ for $0 < x < 1$. The asymptotic behavior $G_0(x) \sim (3/4)x^{-3}$ for large x corresponds to the section of a line which vanishes linearly at some finite distance from the center of the line.

In order to solve (13), it is necessary to find the resolvent $\Gamma_\varepsilon(\tau, \tau')$:

$$S(\tau) = S^*(\tau) + \int_0^\infty \Gamma_\varepsilon(\tau, \xi) S^*(\xi)\, d\xi;$$

here

$$S^*(\tau) = -\frac{3}{8}\left(1 + \frac{3}{2}\varepsilon\right)^{-1} dF(\tau)/d\tau.$$

The resolvent $\Gamma_\varepsilon(\tau, \tau')$ is expressed in terms of the resolvent function $\Phi_\varepsilon(\tau) = \Gamma_\varepsilon(\tau, 0) = \Gamma_\varepsilon(0, \tau)$:

$$\Gamma_\varepsilon(\tau, \tau') = \Phi_\varepsilon(|\tau - \tau'|) + \int_0^{\min\{\tau, \tau'\}} \Phi_\varepsilon(\tau - t)\, \Phi_\varepsilon(\tau' - t)\, dt,$$

which, in turn, satisfies the integral equation

$$\Phi_\varepsilon(\tau) = \frac{1}{2}\int_0^\infty K_\varepsilon(|\tau - \xi|)\, \Phi_\varepsilon(\xi)\, d\xi + \frac{1}{2} K_\varepsilon(\tau). \tag{16}$$

The Laplace transform of $\Phi_\varepsilon(\tau)$ leads to the well-known H-function of Chandrasekhar:

$$H_\varepsilon(x) = 1 + \int_0^\infty \Phi_\varepsilon(t)\, e^{-t/x} dt, \tag{17}$$

which satisfies the nonlinear integral equation

$$H_\varepsilon(x) = 1 + \frac{1}{2} x H_\varepsilon(x) \int_0^\infty \frac{H_\varepsilon(\xi)\, G_\varepsilon(\xi)}{x + \xi}\, d\xi. \tag{18}$$

This equation can also be represented in the form

$$H_\varepsilon(x) = \frac{2}{\int_0^\infty \frac{\xi}{\xi + x} H_\varepsilon(\xi)\, G_\varepsilon(\xi)\, d\xi}. \tag{19}$$

Later we will need the function $P_\varepsilon(\tau, x)$, which satisfies the integral equation

$$P_\varepsilon(\tau, x) = \frac{1}{2}\int_0^\infty K_\varepsilon(|\tau - t|)\, P_\varepsilon(t, x)\, dt + e^{-\tau/x}. \tag{20}$$

It is expressed in terms of $H_\varepsilon(x)$ and $\Phi_\varepsilon(\tau)$:

$$P_\varepsilon(\tau, x) = H_\varepsilon(x)\left\{ e^{-\tau/x} + \int_0^\tau \Phi_\varepsilon(t) \exp[-(\tau - t)/x]\, dt \right\}. \tag{21}$$

Thus, the solution of (13) is equivalent to finding the H-function $H_\varepsilon(x)$ satisfying (18). It is seen from (18) and (19) that $H_\varepsilon(x)$ is monotonically increasing in the interval $0 \leqslant x < \infty$ from $H_\varepsilon(0) = 1$ to $H_\varepsilon(\infty) = \infty$, and

$$\int_0^\infty H_\varepsilon(x)\, G_\varepsilon(x)\, dx = 2. \tag{22}$$

It can also be shown that if $\varepsilon_2 \gg \varepsilon_1$, then $H_{\varepsilon_2}(x) \gg H_{\varepsilon_1}(x)$. It is easy to show that for $x \gg \varepsilon^{-1}$,

$$H_\varepsilon(x) \sim \frac{2x}{\int_0^{1/\varepsilon} \xi H_\varepsilon(\xi)\, G_\varepsilon(\xi)\, d\xi} = \frac{2}{\alpha_1(\varepsilon)}\, x.$$

The first moment of the H-function $\alpha_1(\varepsilon)$ weakly depends on ε; for $\varepsilon \ll 1$, it can be estimated numerically by letting $G_\varepsilon(x) = G_0(x)$ and $H_\varepsilon(x) = H_0(x)$ in the interval $0 < x < \varepsilon^{-1}$. Such estimates show that α_1 (0.01) $\approx$ 3.4, and α_1 (0.0001) $\simeq$ 4.3. For $1 \ll x \ll \varepsilon^{-1}$, the function $H_\varepsilon(x) \sim (2/\sqrt{3})x/\sqrt{\ln x}$.

Let us go into the details of the two concrete cases of the distribution of radiative sources of energy $dF(\tau)/d\tau$ in an atmosphere.

a) Constant Flux of Radiative Energy, F = const. In this case (13) is homogeneous. Differentiating it with respect to τ and comparing the result with (16), we conclude that $dS(\tau)/d\tau = S(0)\Phi_\varepsilon(\tau)$, whence

$$S(\tau) = L(\tau) = S(0)\left[1 + \int_0^\tau \Phi_\varepsilon(t)\, dt\right]. \tag{23}$$

The intensity of the emergent radiation has the form

$$I(0, \mu) = \frac{3}{4}\left(1 + \frac{3}{2}\varepsilon\right)^{-1} \frac{F}{\alpha_1(\varepsilon)} H_\varepsilon\left(\frac{\mu}{1 - \mu^2 + \varepsilon}\right). \tag{24}$$

b) Exponential Distribution of Primary Sources, $F(\tau) = F(0)\, e^{-\tau/\tau_0}$. In this case, comparing (13) with (20), we conclude that

$$S(\tau) = \frac{3}{8}\left(1 + \frac{3}{2}\varepsilon\right)^{-1} F(0)\, \frac{H_\varepsilon(\tau_0)}{\tau_0}\left[e^{-\tau/\tau_0} + \int_0^\tau \Phi_\varepsilon(t) \exp\left(-\frac{\tau - t}{\tau_0}\right) dt\right]. \tag{25}$$

The intensity of the emergent radiation is given by the expression

$$I(0, \mu) = \frac{3}{8}\left(1 + \frac{3}{2}\varepsilon\right)^{-1} F(0)\, \frac{H_\varepsilon(\tau_0)\, H_\varepsilon[\mu/(1 - \mu^2 + \varepsilon)]}{\mu/(1 - \mu^2 + \varepsilon) + \tau_0}. \tag{26}$$

It can be seen from (24) that the angular distribution of the emergent radiation for $\varepsilon \ll 1$ is essentially anisotropic; the direction diagram is of the type of a "pencil." The angular half-width of the "pencil" is $\theta_0 \sim \sqrt{\varepsilon}$. If we let $\varepsilon \to 0$ in (24) for a fixed flux F, then the direction diagram has the form $I(0, \mu) = 1/2 F\delta(1-\mu)$. On the other hand, the expression (24) is the limiting case of the diagram (26) as $\tau_0 \to \infty$. In Fig. 1 the dashed line represents the limiting form of the diagram (24) as $\varepsilon \to 0$ and $I(0, 1)$ = const, which at the same time is the limiting form of (26) for $\varepsilon = 0$ and $\tau_0 \to \infty$. In the same figure for comparison is given the index of emergent radiation in the case of monochromatic conservative scattering when the flux of radiant energy is constant in the atmosphere. In this well-studied case, $I_s(0, \mu) = (\sqrt{3}/4) F H_s(\mu)$, where πF is the constant energy flux and $H_s(\mu)$ is the Ambartsumyan function.

In the case of an exponential distribution of sources of energy the direction diagram of the emergent radiation (26) depends on two parameters: ε and τ_0. For $\tau_0 \ll \varepsilon^{-1}$ the form of the diagram depends only on τ_0; the angular half-width of the "pencil" is $\theta_0 \sim 1/\sqrt{\tau_0}$. The indicatrix of emergent radiation of $\varepsilon = 0$ and $\tau_0 = 20$* is represented in Fig. 1 by a

*The value τ_0 = 20 corresponds to a free path of $\sim 50\ g/cm^2$, in which fast protons, moving in a plasma along the direction of the magnetic field, are stopped at the expense of nuclear collisions; for details, see [2].

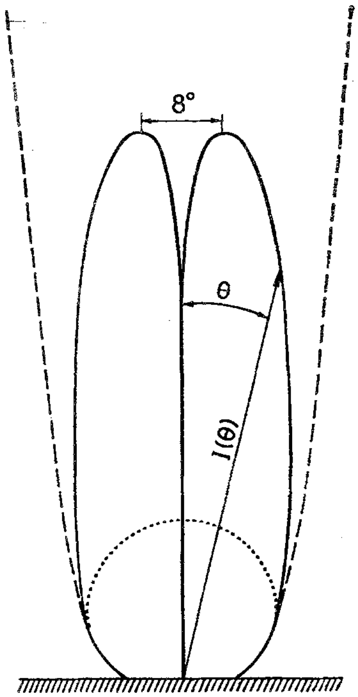

Fig. 1. The indicatrix of the emergent radiation intensity (integrated over the frequency) of a purely scattering atmosphere with a scattering cross section (2). The solid line is the exponential distribution of the primary sources. The dashed line is the constant flux of radiant energy. The dotted line is the constant flux of radiant energy in an atmosphere with an isotropic scattering cross section.

solid line. At the other limit $\tau_0 \gg \varepsilon^{-1}$ the indicatrix (26) is transformed into the indicatrix (24) previously discussed.

4. The Limiting Case of a Strong Field, $\nu/\nu_H \to 0$

The limit passage $\varepsilon \to 0$ does not lead to the appearance of a singularity either in the basic integral equation (13) or in the later calculations. Therefore, the case $\varepsilon = 0$ is of interest, so that we will go into the details of it.

The asymptotics of $G_0(x)$ and $K_0(\tau)$ were described previously. The function $H_0(x)$ is monotonically increasing from $H_0(0) = 1$ to $H_0(\infty) = \infty$. For $x \gg 1$ the asymptotic behavior of $H_0(x)$ has the form*

$$H_0(x) \sim \frac{2}{\sqrt{3}} \frac{x}{\sqrt{\ln x}}. \tag{27}$$

In fact, we will seek an asymptotic of $H_0(x)$ in the form $Cx/\sqrt{\ln x}$. To do this we estimate the integral in (19):

$$\int_0^\infty \frac{\xi}{\xi + x} H_0(\xi) G_0(\xi) d\xi = \int_0^1 \frac{\xi H_0 G_0}{\xi + x} d\xi + \int_1^\infty \frac{\xi H_0 G_0}{\xi + x} d\xi.$$

*The author expresses appreciation to V. V. Ivanov who pointed out the existence of the given asymptotics and proposed its brief derivation presented below.

TABLE 1

ln (20 x)	x	$H_0(x)$	ln (20 x)	x	$H_0(x)$
0	0.05	1.144	5.0	7.421	7.69
0.2	0.06107	1.168	5.2	9.064	8.86
0.4	0.07459	1.196	5.4	11.07	10.2
0.6	0.09111	1.229	5.6	13.52	11.9
0.8	0.1113	1.268	5.8	16.51	13.8
1.0	0.1359	1.312	6.0	20.17	16.1
1.2	0.1660	1.364	6.2	24.64	18.9
1.4	0.2028	1.424	6.4	30.09	22.1
1.6	0.2477	1.493	6.6	36.75	26.0
1.8	0.3025	1.574	6.8	44.89	30.6
2.0	0.3695	1.668	7.0	54.83	36.2
2.2	0.4513	1.776	7.2	66.97	42.8
2.4	0.5512	1.903	7.4	81.80	50.7
2.6	0.6732	2.050	7.6	99.91	60.2
2.8	0.8222	2.221	7.8	122.0	71.5
3.0	1.004	2.42	8.0	149.0	85.1
3.2	1.227	2.65	8.2	182.0	101
3.4	1.498	2.92	8.4	222.4	121
3.6	1.830	3.24	8.6	271.6	144
3.8	2.235	3.61	8.8	331.7	172
4.0	2.730	4.05	9.0	405.2	206
4.2	3.334	4.56	9.2	494.9	247
4.4	4.073	5.16	9.4	604.4	296
4.6	4.974	5.87	9.6	738.2	355
4.8	6.076	6.71	9.8	901.7	425
			10.0	1101	510

It is clear at once that as $x \to \infty, \int_0^1 \frac{\xi H_0 G_0}{\xi + x} d\xi \sim \frac{1}{x}$, while

$$\int_1^\infty \frac{\xi_0 H_0 G_0}{\xi + x} d\xi \sim \frac{3C}{2} \int_1^\infty \frac{d\sqrt{\ln \xi}}{\xi + x} = \frac{3C}{2} \int_1^\infty \frac{\sqrt{\ln \xi}}{(x+\xi)^2} d\xi = \frac{3C}{2x} \int_{1/x}^\infty \frac{\sqrt{\ln x + \ln t}}{(1+t)^2} dt \sim \frac{3C}{2} \frac{\sqrt{\ln x}}{x}.$$

Substituting this result into (19), we can verify (27). We note that (27) implies that $\lim_{\varepsilon \to 0} \alpha_1(\varepsilon) = +\infty$.

The function $H_0(x)$ was computed numerically. To do this the integral representation [4] was used:

$$H_0(x) = \exp\left\{-\frac{1}{\pi}\int_0^\infty \frac{\ln[1 - V_0(\xi/x)]}{1+\xi^2} d\xi\right\},$$

where

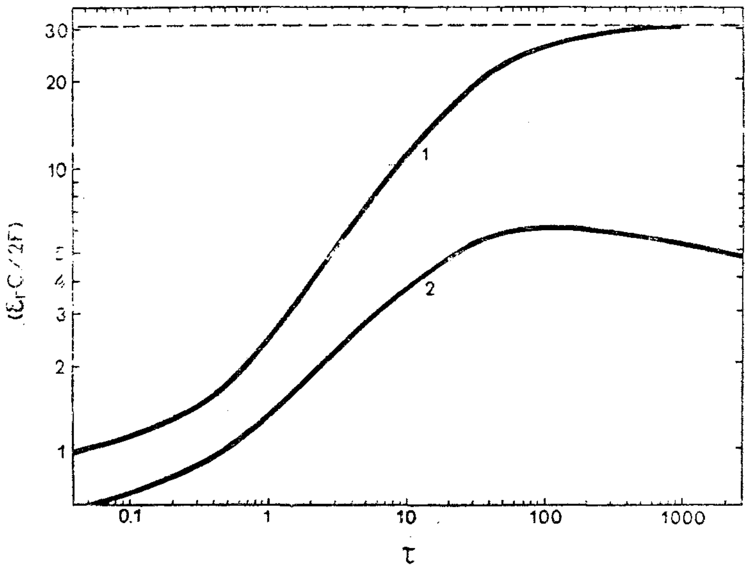

Fig. 2. The density of the radiant energy $\varepsilon_r(\tau)$ (normalized by $2\pi F/c$) as a function of the optical depth for an exponential distribution of primary sources in the two limiting cases: 1) isotropic scattering cross section; 2) scattering cross section of the form (2).

$$V_0(u) = \int_0^\infty \frac{G_0(\xi)}{1 + u^2\xi^2}\, d\xi.$$

The computed values of $H_0(x)$ in a wide interval of variation of x are given in Table 1. In the interval $5 \lesssim x \lesssim 10^4$ the results of the calculations can be given to within ∿ 1% in the form of the approximating formula

$$H_0(x) \simeq 1.625\, x\, (\ln x)^{-0.645}. \tag{28}$$

For comparison we give the asymptotics of the well-known H-functions:

$$H_s(x) \sim \sqrt{3}\, x; \quad H_D(x) \sim 2\pi^{-1/4} x^{1/2} (\ln x)^{1/4}; \quad H_L(x) \sim \left(\frac{9}{2}\right)^{1/4} x^{1/4}.$$

Since $\lim_{x \to \infty} H_0(x)/x = 0$, there is a distinctive qualitative peculiarity of the direction diagram (26): in the center of the "pencil" there is a "hole," since $\lim_{\mu \to 1} I(0, \mu) = 0$ (see Fig. 1). The half-width of the "hole" is $\theta_h \sim 1/\sqrt{x_h}$, where $H_0(x_h)/x_h \sim (1/2) H_0(\tau)/\tau_0$. Numerical calculations show that the "hole" is sufficiently narrow: for $\tau_0 = 20$, its half-width is $\theta_h \sim 30'$.

Along with the directed emergent radiation, of considerable interest is the density distribution of radiant energy $\varepsilon_r(\tau)$ in the atmosphere. The density of the radiant energy $\varepsilon_r(\tau) = (4\pi/c) J(\tau)$, where $J(\tau) = \frac{1}{2}\int_{-1}^{+1} I(\tau, \mu)\, d\mu$ is the average angular intensity. However, unlike ordinary monochromatic scattering, in the present case there is no local connection between $J(\tau)$ and $S(\tau)$; the function $S(\tau)$ is locally connected only with $L(\tau)$ — the intensity, averaged over angles with weight $(1 - \mu^2) = \sin^2\theta$. If we restrict $\varepsilon_r(\tau)$ by rough estimates, we let $\varepsilon_r(\tau) \simeq (4\pi/c) L(\tau)$ (the analog of the Eddington approximation). We note that if $I(\tau, \mu)$ is isotropic, then $J(\tau) = L(\tau)$; on the other hand, numerical estimates for $I(0, \mu)$ of the form

(26) for $\tau_0 = 20$ show that at the outer boundary of the atmosphere $\tau = 0$, where the radiation field has maximal anisotropy, L(0) differs from J(0) by 25%.

The function $L(\tau) = S(\tau) + (3/8)dF(\tau)/d\tau$ is expressed by the resolvent function $\Phi_0(\tau)$: a) for a constant flux $F(\tau) = F$,

$$L(\tau) = L(0)\left[1 + \int_0^\tau \Phi_0(t)\,dt\right]; \tag{29}$$

b) for an exponential distribution of energy sources $F(\tau) = Fe^{-\tau/\tau_0}$,

$$L(\tau) = \frac{3}{8}\frac{F}{\tau_0}\left\{[H_0(\tau_0) - 1]e^{-\tau/\tau_0} + H_0(\tau_0)\int_0^\tau \Phi_0(t)\exp\left(-\frac{\tau - t}{\tau_0}\right)dt\right\}. \tag{30}$$

The expressions (29) and (30) are a special case of (23) and (25). It is easy to show from (27) and (17) that for $\tau \gg 1$

$$\Phi_0(\tau) \sim \frac{2}{\sqrt{3}\ln\tau}. \tag{31}$$

Hence, it follows that for an exponential distribution of primary sources $\lim_{\tau\to\infty}\varepsilon_r(\tau) = \lim_{\tau\to\infty}L(\tau) = 0$. For comparison, we recall that in the case of ordinary monochromatic conservative scattering

$$\lim_{\tau\to\infty} J(\tau) = \frac{\sqrt{3}}{4}FH_s(\tau_0).$$

For numerical estimates of $\varepsilon_r(\tau)$, in place of (29) and (30) it is more convenient to use simpler approximate expressions based on the relation

$$I(0, \mu) = \frac{1-\mu^2}{\mu}\int_0^\infty S(t)\exp\left[-\frac{t}{\mu}(1-\mu^2)\right]dt \simeq S\left(\frac{\mu}{1-\mu^2}\right). \tag{32}$$

We note that (32) is exact equality in the case when $S(\tau) = a + b\tau$. Thus, we are justified to expect that (32) gives a good approximation to reality there where $S(\tau)$ is little different from a linear function. From (32) and (24), (26) we have at once

$$L(\tau) \simeq \begin{cases} L(0)H_0(\tau), & \text{when } F(\tau) = \text{const}; \\ \frac{3}{8}F\left[\frac{H_0(\tau_0)H_0(\tau)}{\tau + \tau_0} - \frac{e^{-\tau/\tau_0}}{\tau_0}\right], & \text{when } F(\tau) = Fe^{-\tau/\tau_0}. \end{cases} \tag{33}$$

The function $\varepsilon_r(\tau)$, calculated according to (33) for the case $F(\tau) = Fe^{-\tau/20}$, is noted in Fig. 2. For comparison we also present the distribution of the density of radiant energy in the case of ordinary conservative scattering, calculated by an expression similar to (33). It can be seen in Fig. 2 that the approximation (32) must also be very good for $\tau \ll \tau_0$ and for $\tau \gg \tau_0$.

We note that the approximation (33) is equivalent to the following qualitative picture of the transfer of radiation in an atmosphere with cross sections of the form (1)-(2): an observer who looks into the atmosphere at an angle θ effectively sees a layer $\tau \sim \cos\theta/\sin^2\theta$, and, consequently, the intensity of radiation $I(0, \mu)$ perceptible by him is proportional to the density of the radiant energy at this depth.

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IONIZATION ZONES AROUND FLARE STARS

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We consider the formation of stationary ionization zones by means of x rays emitted during flares on UV Ceti-type stars. We estimate the possibility of using contemporary equipment for detecting the luminescence of the HII zones in hydrogen lines.

The search for ionization zones around UV Ceti-type flare stars has been a matter of great interest in recent years. The reason is that from the luminescence of these zones in the hydrogen lines we can form an opinion about the flare mechanism and how their energy is released in the form of hard radiation and energetic particles. Although at the present time such zones have not been detected around the nearby flare stars [1-3], nevertheless, in order to clarify the possibility of observing them, it is highly desirable to calculate their physical parameters using current data on the interstellar medium. A similar problem was first considered by Lortet-Zuckermann [4]. She assumed that the luminescence of the circumstellar material in the H_α line of hydrogen is caused by a stream of protons with energies between 10^{-2} and 10 MeV. The protons were assumed to have been accelerated in a flare. In this work, the effect of the extended outer layers of the star (the chromosphere and corona) was not taken into account, although estimates show that the energy losses by protons in the above energy range can be considerable, due to collisions in the hot plasma of the stellar corona. In the present paper, we shall start with the analogy between solar flares and the explosive processes in UV Ceti stars [5, 6], and we shall assume that the fundamental factor in the ionization is x radiation.

At the present time, no bursts of soft x rays have been observed from flare stars. Existing equipment in this spectral range is capable of recording a flux of radiation which is some 10^4 times greater than the flux in the optical region of the spectrum. The true fluxes of x-ray quanta from stellar flares are apparently lower than the threshold of the detectors. Recent reports have pointed out a possible connection between the γ-ray bursts detected by the Vela and IMP-6 satellites [7] and stellar flares. The mean energy flux in an individual γ-ray burst is $\sim 10^{-5}$ erg/cm². This leads to a reasonable value for the total energy of a flare on the star UV Ceti ≡ L-726-8 $\sim 10^{33}$ ergs. (The distance to this star is estimated to be 2.7 pc.) However, if the mean distance to other flare stars is assumed to be of order 100 pc, then the energy released at the source reaches values of 10^{39}-10^{42} ergs. This contradicts the hypothesis that flares of UV Ceti-type stars are electromagnetic in nature. Brecher and Morrison [8] circumvent this difficulty by reckoning that in the γ-ray range the radiation from stellar flares is highly anisotropic and is generated in enormous magnetic structures similar to streamers in the solar corona. The work of Karitskaya [9] was also devoted to an interpretation of the γ-ray bursts in terms of stellar flares.

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