THERMALIZATION LENGTH OF RESONANCE RADIATION IN THE CASE OF PARTIAL FREQUENCY REDISTRIBUTION

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A study is made of the propagation of resonance radiation in a homogeneous infinite medium in the case when the frequency redistribution is due to natural line broadening and the Doppler shift resulting from the thermal motion of the scattering atoms. The Monte Carlo method is used to calculate the mean displacement $\langle r(t) \rangle$ and the mean number of scatterings $\mathrm{I\!I}\langle (t) \rangle$ undergone by resonance photons in a purely scattering medium during time t. It is shown that in the Lorentz wings of the Voigt profile the propagation of the radiation in space and the evolution of the line profile both have a diffusion nature. The part played by the recoil effect in resonance scattering is discussed.

1. Introduction

Hitherto, radiative transfer in the frequencies of spectral lines has been investigated analytically only under the assumption of complete frequency redistribution in each resonance scattering event and only in the simplest cases of a homogeneous medium with spherical or flat geometry [1]. In more complicated situations, it is necessary to solve the transfer equations numerically on each occasion or be content with simple order-of-magnitude estimates. In the latter case, an important part is played by the concept of the thermalization length τ_{th}, which is the average distance between the positions of creation and destruction of a photon expressed in mean free paths. It is important to know this quantity in the cases when the cross sections of the processes of "true" absorption (leading to annihilation of the photons) are many times smaller than the resonance scattering cross section.

Usually expressions for the thermalization length are derived on the basis of an asymptotic expansion of the exact solutions of the transfer equation. This was the procedure used to obtain the well-known expressions for the thermalization length τ_{th} of resonance lines in the case when the hypothesis of complete redistribution can be used [1]. However, in astrophysics one encounters situations for which this hypothesis is certainly incorrect [2], and then it is particularly important to be able to estimate τ_{th}, since the investigation of the exact solutions of the transfer equation even in the framework of the simplest geometries leads to insuperable mathematical difficulties [3]. In our earlier [4] we proposed a method that makes it possible to estimate the thermalization length of resonance lines in some cases when it is not possible to solve the complicated equation for the propagation of the radiation in space. The method reduces to analyzing the behavior of the spectral profile of a line in an infinite homogeneous medium filled with an isotropic radiation field with subsequent averaging of the expressions for the displacement of monochromatic photons through the known profile. The method was used to obtain expressions for τ_{th} in the case when the line has a finite natural width and the frequency redistribution is due to the Doppler displacement resulting from scattering by atoms possessing a Maxwellian velocity distribution.

Strictly speaking, however, the proposed method of estimating τ_{th} is not justified mathematically, and doubts remain as to whether it gives the correct asymptotic behavior of the exact solution of the corresponding kinetic equation. Moreover, it does not enable one to establish the correct numerical coefficients of the corresponding asymptotic expressions for τ_{th} or investigate the transition region between the two asymptotic behaviors. To eliminate all these shortcomings, we have undertaken a series of numerical

Institute of Theoretical and Experimental Physics. Translated from Astrofizika, Vol. 17, No. 1, pp. 125-139, January-March, 1981. Original article submitted February 27, 1980; accepted for publication May 20, 1980.

0571-7132/81/1701-0069$07.50

Monte Carlo calculations, the results of which are presented in the second section of the present paper. These calculations have completely confirmed the functional dependence of τ_{th} established in [4], though the differences in some numerical coefficients reach a factor ~3. The validity of the investigated asymptotic expressions for τ_{th} was also recently confirmed by a direct analysis of the asymptotic behavior of the solutions of the stationary transfer equation [5, 6].

In the third section of the paper, we discuss the part played by recoil in resonance scattering, which was not taken into account earlier in [4].

2. Formulation of the Problem

We first give a rigorous definition of the physical quantities whose values were found in the numerical calculations. It is convenient to introduce the following dimensionless quantities: the frequency $x = (\nu - \nu_0)/\Delta\nu_D$, the time $\bar{t} = tcN\Sigma$, the radius vector $\bar{r} = rN\Sigma$, the cross section $\bar{\sigma}(x) = \sigma(x)/\Sigma$ of scattering in the line, and the velocity $\bar{v} = v/(2kT/M)^{1/2}$ of the scattering atoms. Here, ν_0 is the frequency of the center of the line profile, $\Delta\nu_D = \nu_0(2kT/Mc^2)^{1/2}$ is the Doppler width of the line, M is the mass of the scattering atoms, and N (cm^{-3}) is their density. The quantity

$$\Sigma \equiv \int\limits_{-\infty}^{+\infty} \sigma(x)\,dx = \frac{\pi e^2}{m_e c}\,\frac{f_{12}}{\Delta\nu_D} \tag{1}$$

has the dimensions of a cross section and is determined by the oscillator strength f_{12} of the resonance transition 1 → 2. The dimensionless coordinates $\bar{r}$ and the time $\bar{t}$ are chosen in such a way that light traverses the distance $|\Delta\bar{r}| = \Delta\bar{t}$ in the time $\Delta\bar{t}$. In all that follows, we use dimensionless units and omit the bar.

In the general case, the kinetic equation describing the evolution of the field of resonance radiation in the homogeneous infinite medium is

$$\frac{\partial n}{\partial t} + \mathbf{\Omega}\cdot\frac{\partial n}{\partial \mathbf{r}} = -[\beta + \sigma(x)]\,n + \lambda \int\limits_{-\infty}^{+\infty}\int\limits_{4\pi} n(t, x', \mathbf{r}, \mathbf{\Omega}')\,\frac{d^2\sigma(x' \to x, \mu)}{dx'\,d\mathbf{\Omega}'}\,d\mathbf{\Omega}'\,dx'. \tag{2}$$

Here, $n(t, x, \mathbf{r}, \mathbf{\Omega})\,dx\,d\Omega$ (photon/cm^3) is the volume density of the photons in the frequency interval (x, x + dx) propagating in the direction of the unit vector $\mathbf{\Omega}$ in the solid angle $d\Omega$; β is the cross section of absorption in the continuum normalized by Σ; λ is the probability of a photon's surviving in a resonance scattering event (the albedo of single scattering); $\mu \equiv \mathbf{\Omega}\cdot\mathbf{\Omega}'$. Equation (2) has been written down with neglect of induced scattering processes and under the assumption that within the line profile $|x\Delta\nu_D| \ll \nu_0$. It is also assumed that the radiation field does not change during a time of the order of the lifetime of the excited state and only the photon delay in flight is taken into account.

The differential cross section of resonance scattering used in the present paper,

$$\frac{d^2\sigma(x \to x', \mu)}{dx'\,d\mathbf{\Omega}'} = \frac{1}{4\pi^{7/2}}\iiint\limits_{-\infty}^{+\infty} \frac{a\exp(-v^2)}{a^2 + (x - \mathbf{v}\cdot\mathbf{\Omega})^2}\,\delta[x' - x - \mathbf{v}\cdot(\mathbf{\Omega}' - \mathbf{\Omega}) + \varepsilon(1 - \mu)]\,d^3v \tag{3}$$

is obtained under the assumption that in the rest frame of the scatterer the scattering is isotropic with differential cross section

$$\frac{d^2\sigma(x \to x')}{dx'\,d\mathbf{\Omega}'} = \frac{1}{4\pi^2}\,\frac{a}{a^2 + x^2}\,\delta(x - x'), \tag{4}$$

where the damping constant $a = \Gamma/4\pi\Delta\nu_D$ is the ratio of the natural width of the line to twice the Doppler width. The assumption of isotropic scattering hardly affects the values of the thermalization length, and the results of calculations with Rayleigh phase function $d\sigma/d\Omega' \sim 3/8(1 + \mu^2)$, which is characteristic of dipole transitions, are, to within the statistical errors, identical to the results obtained for isotropic scattering. In the expression for the change of frequency on scattering

$$x' - x = \mathbf{v}\cdot(\mathbf{\Omega}' - \mathbf{\Omega}) - \varepsilon(1 - \mu) \tag{5}$$

allowance is made for only the Doppler effect in the first order in v/c and the recoil effect, whose role will be discussed in Sec. 3. In Secs. 1 and 2, the recoil effect is ignored, and the constant

$$\varepsilon = h\nu_0/(2kTMc^2)^{1/2} \ll 1 \tag{6}$$

is taken equal to zero. The neglect of the effects leading to a frequency shift $\sim v^2/c^2 \sim kT/Mc^2$ (the Doppler shift $\sim v^2/c^2$, aberration on the transition from the rest frame of the atom to the laboratory frame, noninvariance of the cross section) is justified by the circumstance that in the physically interesting situations the effective production and scattering of resonance photons occurs subject to the condition $kT \lesssim 0.1h\nu_0$, when the degree of ionization of the scattering atoms is not too high.

The concept of the thermalization length is meaningful only when some mechanism of "true" absorption operates besides the resonance scattering. Each concrete process of "true" absorption is one of two types: (A) weak absorption in the continuum with cross section $\beta \ll 1$ independent of the frequency; (B) weak violation of the conservative nature of the resonance scattering, when the albedo of single scattering satisfies $\lambda > 1$ but $1 - \lambda \ll 1$. In case (A), the thermalization length τ_{th} is defined naturally as the mean displacement of the photons in the stationary problem:

$$\tau_{th} = \frac{\int\limits_{-\infty}^{+\infty}\iiint\int\limits_{4\pi} r n(x, \mathbf{r}, \mathbf{\Omega})\, d\Omega dx d^3r}{\int\limits_{-\infty}^{+\infty}\iiint\int\limits_{4\pi} n(x, \mathbf{r}, \mathbf{\Omega})\, d\Omega dx d^3r}. \tag{7}$$

Here, $n(x, \mathbf{r}, \mathbf{\Omega})$ is the solution of Eq. (2) in the stationary case $\partial n/\partial t \equiv 0$ for $\lambda = 1$ and one of the boundary conditions

$$n(x, \mathbf{r}, \mathbf{\Omega})\big|_{r\to 0} = \frac{S}{4\pi r^2}\,\delta(\mathbf{\Omega} - \mathbf{r}/r)\begin{Bmatrix}\delta(x)\\ \sigma(x)\end{Bmatrix}, \qquad \begin{matrix}(8a)\\(8b)\end{matrix}$$

where S is the power of the photon source at the coordinate origin, and $r \equiv |\mathbf{r}|$. The condition (8a) describes the situation when the photons are produced only at the line center $x = 0$, whereas under the condition (8b) they are initially distributed over the frequency with probability density $\sigma(x)$. The physically most interesting case is b), which corresponds to real processes of excitation of resonance transitions (electron collision, recombination, etc.). However, case a) is more amenable to investigation and leads to a more rapid establishment of the asymptotic regime.

In case (B) when $\beta = 0$ and $\lambda < 1$, the definition of the thermalization length (7) becomes meaningless, since, as is readily seen, the corresponding integrals diverge. Indeed, under the boundary condition (8b) the fraction of the radiation emitted in the frequency interval $(x, x + dx)$ is proportional to $\sigma(x)$, the displacement of the photons before the first scattering is $\sim\sigma^{-1}(x)$, and the mean displacement calculated in accordance with (7) is infinite. Bearing in mind this last circumstance, and also a number of other considerations, we shall attempt to determine the thermalization length in a different way. First, we determine the mean displacement $\langle r(t)\rangle$ and the mean number of scatterings $\langle \Pi(t)\rangle$ undergone by photons during time t in the case of conservative scattering:

$$\langle r(t)\rangle = \int\limits_{-\infty}^{+\infty}\iiint\int\limits_{4\pi} r n(t, x, \mathbf{r}, \mathbf{\Omega})\, d\Omega dx d^3r, \tag{9a}$$

$$\langle \Pi(t)\rangle = \int\limits_{0}^{t}\int\limits_{-\infty}^{+\infty}\iiint\int\limits_{4\pi} \sigma(x) n(t, x, \mathbf{r}, \mathbf{\Omega})\, d\Omega dx d^3r dt, \tag{9b}$$

where $n(t, x, \mathbf{r}, \mathbf{\Omega})$ is the solution of Eq. (2) for $\beta = 0$, $\lambda = 1$ and one of the initial conditions

$$n(t, x, \mathbf{r}, \mathbf{\Omega})\big|_{t=0} = \frac{\delta(r - 0)}{4\pi r^2}\,\delta(\mathbf{\Omega} - \mathbf{r}/r)\begin{cases}\delta(x)\\ \sigma(x)\end{cases} \qquad \text{(10a)}\ \text{(10b)}$$

The physical meaning of the initial conditions (10) is as follows: at time t = 0, one photon is emitted from the coordinate origin in the line center (or with profile σ(x)), and all directions of emission are equally probable.

On the basis of the concepts $\langle r(t) \rangle$ and $\langle \Pi(t) \rangle$ introduced above, we give the following definitions of the thermalization length: in case (A)

$$\tau_{th} = \langle r(\beta^{-1}) \rangle ; \qquad (11)$$

in case (B)

$$\begin{cases}\tau_{th} = \langle r(t_{th}) \rangle . & (12a)\\ \langle \Pi(t_{th}) \rangle = (1 - \lambda)^{-1}. & (12b)\end{cases}$$

The physical meaning of (11) is that the thermalization length is the mean displacement of a photon during time t = β^{-1}, which is equal to the mean lifetime of a photon in a medium with absorption cross section in the continuum β. It is readily seen that the definitions (7) and (11) are equivalent in their physical meaning and can differ by not more than a numerical factor whose value is near 1, since the law of variation of $\langle r(t) \rangle$ differs weakly from the law $\langle r(t) \rangle \sim t$ (see below), for which the numerical factor is exactly 1. This conclusion is confirmed by the results of direct numerical calculations in accordance with Eqs. (7) and (11), which never differ by more than 10%. Similarly, the thermalization length in the case (B) is in accordance with (12) the mean displacement during the time t_{th}, during which the mean number of scatterings reaches the value $(1 - \lambda)^{-1}$, which in turn is equal to the mean number of scatterings of a photon in an infinite medium for $\beta = 0$ and $0 < \lambda < 1$.

Note that to unify the expressions in the present paper the scattering cross section and the other quantities are, in contrast to [4], normalized by Σ, and not by the absorption cross section at the line center. As a result, to go over from (11) and (12) to the expressions usually adopted for the thermalization length expressed in mean free paths at the center of the profile, it is necessary to take into account the correction factor

$$\varphi(0) = \begin{cases}\pi^{-1/2}, & a \ll 1,\\ (\pi a)^{-1}, & a \gg 1.\end{cases} \qquad (13)$$

3. Results of Calculations

In the case when there is no natural broadening (a = 0), the propagation of line photons in space occurs as follows [7]: the photon is scattered many times near the center of the profile, remaining virtually in the same place until it is shifted randomly in frequency by $|x| \gtrsim [\ln(r\pi^{-1/2})]^{1/2}$, after which it directly covers a distance r in one flight. In this case, as is shown in [4], the asymptotic expressions for $\langle r(t) \rangle$ and $\langle \Pi(t) \rangle$ are

$$\langle r(t) \rangle = \Lambda_D t (\ln t)^{-1}, \qquad (14a)$$

$$\langle \Pi(t) \rangle = \frac{1}{2} t (\ln t)^{-1/2}. \qquad (14b)$$

If the natural width of the resonance line is of the same order as the Doppler width ($a \sim 1$), the propagation of photons in space is of a diffusion nature, and

$$\langle r(t) \rangle = \Lambda_a a^{-1/4} t^{3/4}, \qquad (15a)$$

$$\langle \Pi(t) \rangle = \frac{4}{3}\,\frac{(2\pi)^{1/4}}{\Gamma(1/4)}\,a^{-1/4} t^{3/4} = 0.582\,a^{-1/4} t^{3/4}. \qquad (15b)$$

For $0 < a \ll 1$ and not too large $t < t^*$, the asymptotic regime (14) is first established, and then for $t > t^{**}$ the asymptotic regime (15). Further, as was pointed in [4], the

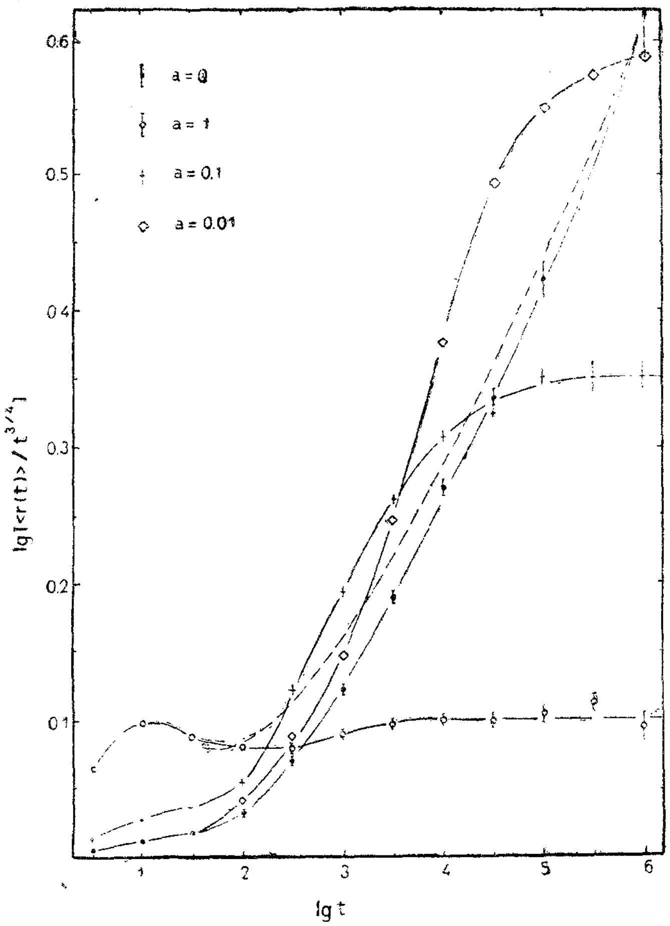

Fig. 1. Mean displacement of photons of the line $\langle r(t) \rangle$ [in units of $(\Sigma N)^{-1}$] in an infinite homogeneous medium during the time t (in units of $(\Sigma Nc)^{-1}$) divided by $t^{3/4}$ with the initial condition (10a). The continuous curves join the values calculated by the Monte Carlo method. The broken curve is the asymptotic behavior (14a) for $\Lambda_D = \sqrt{\pi}$.

regions of applicability of the asymptotic behaviors (14) and (15) are not connected, and t^* may differ strongly from t^{**}. In [4], the values of the coefficients Λ_D and Λ_a were not established.

In Figs. 1 and 2, we show the results of calculations of $\langle r(t) \rangle / t^{3/4}$ for the initial conditions (10a) and (10b), respectively, and in Fig. 3 we give the calculated values of $\langle \Pi(t) \rangle / t^{3/4}$. The continuous curves join the points calculated by the Monte Carlo method, while the broken curves are the graphs of the asymptotic expressions (14) and (15). The indicated limits of the statistical errors correspond to $\pm 1\sigma$. The method of calculation is described in the Appendix.

Comparison of the continuous and the broken curves shows that the expressions (14) and (15) correctly describe the asymptotic behavior of $\langle r(t) \rangle$ and $\langle \Pi(t) \rangle$ as determined by the relations (9). The values of the constants Λ_D and Λ_a, found by the method of least squares from the given calculations for $t \geqslant 10^4$, are

$$\Lambda_D = 1.71 \pm 0.04\ (1\sigma), \quad \Lambda_a = 1.267 \pm 0.007\ (1\sigma). \tag{16}$$

We allow ourselves a conjecture, namely, that $\Lambda_D = \sqrt{\pi}$.

Analyzing the behavior of the curves $\langle r(t) \rangle / t^{3/4}$ for $a = 0.1$ and 0.01 in Figs. 1 and 2, we see immediately that in complete agreement with the conclusion of [4] the transition from the asymptotic behavior (14a) to the asymptotic behavior (15a) in the case $a \ll 1$ does not occur in an entirely usual manner. Indeed, there is a intermediate

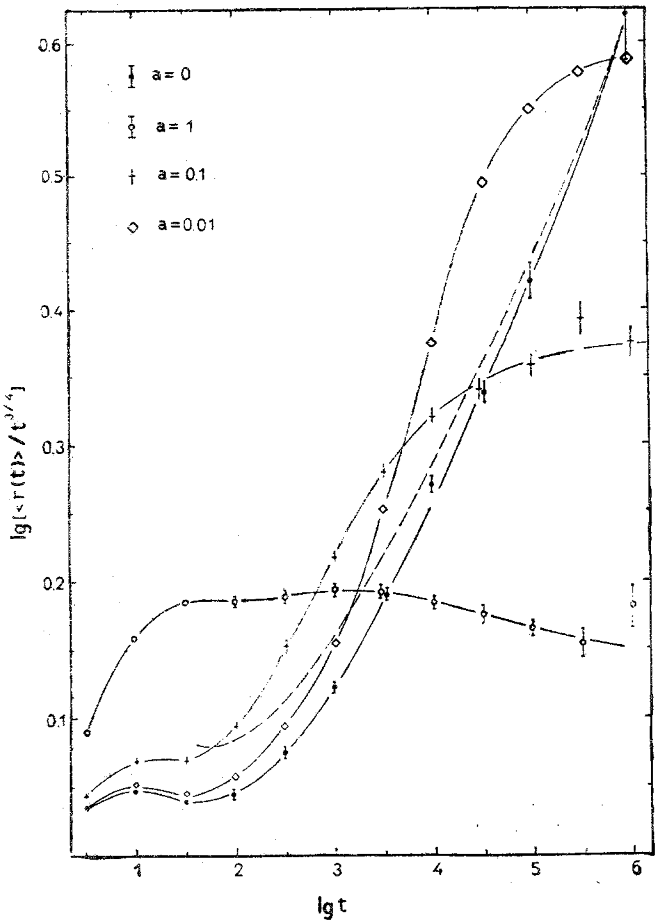

Fig. 2. The same quantities as in Fig. 1 but for the initial condition (10b).

region $t_r^* < t < t_r^{**}$, in which the graph of the function $\langle r(t)\rangle/t^{3/4}$ goes upward (and not downward, as one might expect) from the increasing asymptotic behavior (14a) before tending to the horizontal asymptotic behavior corresponding to the expression (15a). The value of t_r^{**} can be estimated by simply equating the expressions (14a) and (15a), which gives

$$t_r^{**}(\ln t_r^{**})^{-4} = 0.26a^{-1}. \tag{17}$$

To estimate t_r^*, it is convenient to use the following physical argument: formula (14a) must be valid until the half-width of the line profile, which for $a = 0$ increases in accordance with the law $x_m(t) = [\ln(t\pi^{-1/2})]^{1/2}$, exceeds the half-width of the Doppler core $x_D(a)$, which is determined by

$$x_D^2 \exp(-x_D^2) = \pi^{-1/2} a; \tag{18}$$

in other words

$$t_r^*/\ln t_r^* = \pi a^{-1}. \tag{19}$$

For $a = 0.01$, the values of t_r^* and t_r^{**} calculated in accordance with (19) and (17) are, respectively, $2.45 \cdot 10^3$ and $9.26 \cdot 10^5$.

In the case of the mean number of scatterings $\langle \Pi(t)\rangle$, the situation described above simplifies significantly, as can be seen from Fig. 3, and the transition from the asymptotic behavior (14b) to (15b) occurs in the most usual manner at $t \sim t_\Pi^* = t_\Pi^{**}$, where

$$t_\Pi^*(\ln t_\Pi^*)^{-2} = 2\pi\left[\frac{8}{3\Gamma(1/4)}\right]^4 a^{-1} = 1.84a^{-1}. \tag{20}$$

The asymptotic expressions (14) and (15) are valid for both variants of the initial

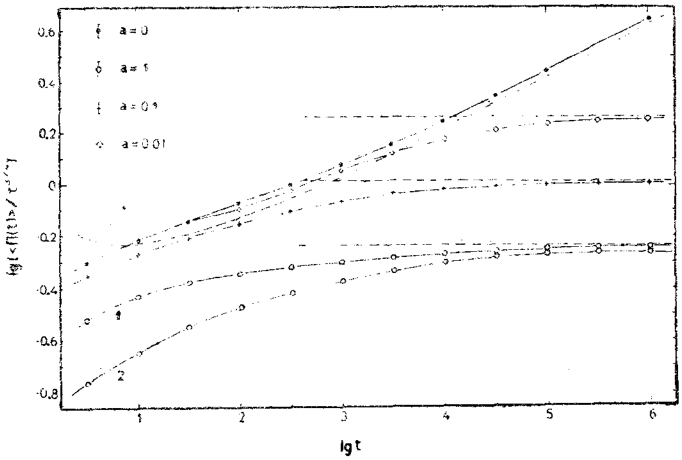

Fig. 3. Mean number of scatterings $\langle \Pi(t) \rangle$ of line photons in an infinite homogeneous medium during the time t divided by $t^{3/4}$. The continuous curves join the values calculated by the Monte Carlo method. The asymptotic behaviors (14b) and (15b) are represented by the broken lines: 1) initial condition (10a), 2) initial condition (10b).

condition (10). However, under the condition (10b) the transition to the corresponding asymptotic behavior can occur much more slowly than under the condition (10a), as happens, for example, for $a = 1$ (cf. Fig. 1 and Fig. 2). It can be seen in Fig. 2 that even at $t = 10^6$ the value of $\langle r \rangle / t^{3/4}$ still significantly ($\gtrsim 10\%$) differs from the asymptotic value (15a). Such behavior can be readily understood on the basis of a simple order-of-magnitude estimate. The fraction of photons initially omitted in the line wings in the frequency interval (x, x + dx) is $\sim (a/\pi x^2)dx$. During the time t, these photons diffuse over a distance $r \sim (t(\pi x^2/a))^{1/2}$, and their contribution to $\langle r(t) \rangle$ reaches

$$\Delta \langle r \rangle = 2\int_{x_m}^{x_1} \frac{a}{\pi x^2}\left(t\,\frac{\pi x^2}{a}\right)^{1/2} dx \sim \frac{2}{\sqrt{\pi}}\,(at)^{1/2}\ln[(at)^{1/4}], \tag{21}$$

where $x_m \approx (at)^{1/4}$ is the half-width of the part of the line profile in which a quasi-equilibrium distribution of the photons with respect to the frequency can be established, its contribution to $\langle r \rangle$ giving the asymptotic term (15a), while $x_1 \approx (at)^{1/2}$ is the half-width of the part of the profile within which the displacement of the photons in space occurs in a diffusion manner. As can be seen from (21), the relative contribution of the region of frequencies $x_m < |x| < x_1$ to the mean displacement,

$$\Delta \langle r \rangle / \langle r \rangle \approx a^{3/4} t^{-1/4} \ln[(at)^{1/4}], \tag{22}$$

decreases rather slowly with increasing t, and at $t = 10^6$ it is ~10%.

4. The Effect of Recoil in Resonance Scattering

The redistribution function has been analyzed with allowance for the recoil effect for the case $a = 0$ by Field [8]. Adams [9] has advanced a number of weighty qualitative arguments indicating that the recoil effect is entirely unimportant for $a = 0$, whereas for $a \neq 0$ it must be taken into account from values of the optical thickness so great that in practice they are never realized. In this section, we derive an equation that describes with allowance for recoil the evolution of the line profile in the wings of the Voight profile; the analysis of this equation not only confirms but actually strengthens the conclusion drawn by Adams.

Suppose the radiation field $n(t, x, \mathbf{r}, \Omega) = n(t, x)$ is isotropic and does not depend on r. In this case, proceeding from the expression (3), we can reduce the redistribution function to the form

$$R(x, x') = \int\limits_{4\pi} \frac{d^2\sigma(x \to x', \mu)}{dx'd\Omega'} d\Omega' = \frac{a}{\pi^{3/2}} \exp(-2\varepsilon u - \varepsilon^2) \int\limits_{-\infty}^{\infty} \frac{\exp(2\varepsilon p)}{a^2 + (p - s - \varepsilon)^2} \int\limits_{|p|+|u|}^{\infty} e^{-y^2} dy dp, \tag{23}$$

where $u = (x' - x)/2$, $s = (x' + x)/2$. Expanding (23) in a series in the small parameters ε and $|s|^{-1}$, we arrive at the expression

$$R(x, x') \approx \frac{1}{\pi^{3/2}} \frac{a}{a^2 + s^2} (1 - 2\varepsilon u) \psi(|u|), \tag{24}$$

which gives a good approximation of the redistribution function (23) in the far wings of the Voight profile; here,

$$\psi(y) = e^{-y^2} - 2y \int\limits_{y}^{\infty} e^{-x^2} dx. \tag{25}$$

Going over by means of (24) to the diffusion approximation, as we did earlier in [4], we obtain from (2) for $\beta = 0$ and $\lambda = 1$

$$\frac{\partial n}{\partial t} = \frac{a}{2\pi} \frac{\partial}{\partial x} \left[\frac{1}{a^2 + x^2} \left(\frac{\partial n}{\partial x} + 2\varepsilon n \right) \right]. \tag{26}$$

The equilibrium solution of Eq. (26) is

$$n(x) = \text{const}\, e^{-2\varepsilon x} = \text{const}\, e^{-h\nu/kT} \tag{27}$$

in complete agreement with Field's results [8] for $a = 0$.*

Comparing (27) with the equilibrium solution n = const of Eq. (26) for $\varepsilon = 0$, we conclude that the recoil effect must be taken into account from the time when the half-width $x_m \approx (at)^{1/4}$ of the broadening line becomes comparable with ε^{-1}. A more detailed investigation of Eq. (26) leads to the same conclusion. Setting $\varepsilon = 0$ and bearing in mind that $|x| \gg a$ in the region in which we are interested, we rewrite (26) in the form

$$\frac{\partial n}{\partial t} = \frac{a}{2\pi} \frac{\partial}{\partial x} \frac{1}{x^2} \frac{\partial n}{\partial x}. \tag{28}$$

The solution of Eq. (28) for the initial condition $n(0, x) = \delta(x - x_0)$ has the form

$$n(t, x, x_0) = \frac{\pi}{4at} \exp\left(-\pi \frac{x^4 + x_0^4}{8at}\right) \left[(x^2 x_0^2)^{3/4} I_{-3/4}\left(\pi \frac{x^2 x_0^2}{4at}\right) + (x x_0)(x^2 x_0^2)^{1/4} I_{3/4}\left(\pi \frac{x^2 x_0^2}{4at}\right) \right], \tag{29}$$

and for $x_0 = 0$ reduces to the expression (24) in [4]. (Above, $I_\nu(y)$ is a Bessel function of imaginary argument.) Making the corresponding calculations, we see that $|\partial n/\partial x|$ becomes comparable with the term $2\varepsilon n$ in the whole of the physically important region of the profile (29) from the time

$$t \sim t_\varepsilon = a^{-1}\varepsilon^{-4} \Rightarrow x_m(t_\varepsilon) \approx \varepsilon^{-1}. \tag{30}$$

But since the condition $x_m \sim \varepsilon^{-1}$ means that the line width becomes ~kT, there is no point in considering times $t \gg t_\varepsilon$, since the condition of applicability $|\Delta\nu| \ll \nu$ of the theory of resonance scattering is then violated.

As an example, we give the values of the basic parameters for the line L_α of hydrogen at 10^4 °K: $a = 4.74 \cdot 10^{-4}$, $\varepsilon = 2.54 \cdot 10^{-4}$, $t_\varepsilon = 5.1 \cdot 10^{17}$. If a photon is to survive for the time t_ε in a plasma cloud, the optical thickness of the cloud must exceed $r \sim a^{-1/4} t_\varepsilon^{3/4} \sim 10^{14}$, which exceeds the analogous estimate of Adams [9] by more than three orders of magnitude and completely confirms the conclusion that the recoil effect plays no part in the cases of practical interest.

5. Conclusions

In this paper, we have concentrated our attention on studying the behavior of the

*To obtain the Planck formula instead of Wien's law (27), it is necessary to take into account not only the recoil effect but also induced scattering and effects $\sim v^2/c^2$, which describe the transfer of the thermal energy of the atoms to the photons.

two quantities that characterize radiative transfer in resonance lines in a purely scattering medium — the mean displacement $\langle r(t) \rangle$ and the mean number of scatterings $\langle \Pi(t) \rangle$ undergone by line photons in the time t. Since the use of these concepts to analyze concrete physical situations is not standard, we wish to emphasize some advantages of their use. Above all, these quantities have a very clear physical meaning, which significantly facilitates operations with them, makes possible a rigorous definition of them [see (9)], and means one can use a simple algorithm to find their numerical values by the Monte Carlo method. In Sec. 2 we have already noted, for example, that the concept of the thermalization length τ_{th} in the case (5), when $\lambda < 1$, does not have these advantages. It is also important that $\langle r(t) \rangle$ and $\langle \Pi(t) \rangle$ together carry more information than say τ_{th} alone. Knowing $\langle r(t) \rangle$ and $\langle \Pi(t) \rangle$, we can readily estimate τ_{th} both in the presence of weak continuum absorption as well as in the case of weak violation of the conservative nature of the scattering, the expressions (11) or (12), respectively, being used. As an example which demonstrates the convenience of a discussion in terms of $\langle r(t) \rangle$ and $\langle \Pi(t) \rangle$, we also point out that (15a) and (15b) directly yield the behavior established in [10] on the basis of rather lengthy numerical calculations, namely, the mean number of scatterings of a photon emitted at the center of a plane-parallel layer with transverse optical thickness $2\tau_0$ at the line center is proportional to τ_0 for $\tau_0 \gg 1$, and the coefficient of proportionality does not depend on the damping constant a.

In conclusion, we recall that in [4] the asymptotic expressions (15) were obtained under the assumption that in the wings of the Voigt profile the photons are displaced both in frequency and the coordinates in a diffusion manner. The agreement between the results of the numerical calculations and the expressions (15) here serves as a kind of justification of the possibility (a priori far from obvious) of using the diffusion approximation with respect to the frequency and the Eddington approximation in space, as was done, for example, in [11].

Appendix

The main block of the program designed to calculate one history from the time of creation of a photon to the end of an interval of time t included the following steps:

a) random selection of the frequency x and the initial direction of emission $\mathbf{\Omega}$;

b) random selection of the mean free path in the direction $\mathbf{\Omega}$ in accordance with the known scattering cross section $\sigma(x)$;

c) random selection of the velocity components $v_{\parallel} = \mathbf{v}\cdot\mathbf{\Omega}$, $v_{\perp} = |\mathbf{v} - v_{\parallel}\mathbf{\Omega}|$ of the scattering angle and the azimuth φ_V of the velocity v in accordance with the densities of the probability distributions:

$$p(v_{\parallel}) = \frac{1}{\sigma(x)} \frac{a}{\pi^{3/2}} \frac{e^{-v_{\parallel}^2}}{a^2 + (x - v_{\parallel})^2}, \qquad -\infty < v_{\parallel} < +\infty;$$

$$p(v_{\perp}) = 2v_{\perp} e^{-v_{\perp}^2}, \qquad 0 < v_{\perp} < +\infty; \quad p(\varphi_V) = \frac{1}{2\pi}, \qquad 0 < \varphi_V < 2\pi;$$

d) random selection of a new direction of emission $\mathbf{\Omega}'$ of the photon and calculation of the new frequency x'; if the history does not terminate, return to step b). In the actual implementation, the mean free path for $a \neq 0$ was selected simultaneously with $v_{\parallel}$ by the fictitious scattering method [12]. To economize computer time, this block of the diagram was written in the BEMSh machine language. The pseudorandom numbers γ_k were chosen by means of the recursive procedure [12]

$$m_0 = 1; \quad m_{k+1} = 5^{17} m_k \pmod{2^{40}}; \quad \gamma_k = 2^{-40} m_k.$$

With a history of duration $t = 10^6$, ten hours of computing time on a BÉSM-6 was sufficient to calculate ~1200 histories for $a = 1$ and ~280 histories for $a = 0.01$.

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COHERENT PROPERTIES OF THE RADIATION OF A ROTATING STAR

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The complex degree of coherence of the radiation of a rotating star in a spectral line is calculated. It is shown that the rotation velocity and orientation axis of a star can be determined by means of modern stellar interferometers.

1. At the present time, there is considerable interest in astronomy in stellar interferometry. In the first place, this is due to the improvement of old and the development of new methods of interferometric observations. Here, we may mention intensity interferometry [1], speckle interferometry [2], amplitude interferometry [3] and there are others as well (see the reviews [4, 5] and the references given there; the most complete information about the present state and prospects for the development of stellar interferometry can be found in [6]). The intensity interferometer at Narrabri (Australia) has already achieved a resolution of $3 \cdot 10^{-4}$ arc second [7], and interferometers under design and construction will have a resolution $\sim 10^{-4}$–10^{-5} arc second at limiting magnitudes 7^m–13^m [4-6, 8]. These observations make tens of thousands of objects accessible, and the possibility is opened up of using the methods of optical interferometry to determine some of the most important parameters of stars (see [4-6, 8, 9]), including data on the rotation, pulsation, and, generally, the motion of the surface of a star. It should be noted that in the literature the influence of rotation of a star on the coherence of its radiation has hitherto been discussed only from the point of view of consequences of rotation such as oblateness of the photosphere, nonuniformity of the brightness distribution over the disk, and so forth [10, 11] (see also the references in [4, 6]). The corresponding effects are small and, as a rule, of the order of the errors of the measurements. The rotation of a star itself has not been considered in the quoted papers.

In the present paper, we analyze the direct influence of the motion of the surface of a rotating star on the spatial coherence of its electromagnetic radiation, and we show that observations in spectral lines with modern interferometers will make it possible to determine parameters of rotating stars such as the radial component of the equatorial velocity and the orientation of the projection of the axis of rotation on the plane perpendicular to the line of sight. The possibility of determining the latter is of particular interest, since, on the one hand, this parameter is effectively inaccessible by other methods and, on the other, knowledge of it would be extremely important in the study of magnetic stars, in the investigation of genetically related star groups and clusters, double and multiple stars, in searches for black holes as the components of multiple systems, and in a number of other astrophysical problems.

Kiev State University. Translated from Astrofizika, Vol. 17, No. 1, pp. 141-153, January-March, 1981. Original article submitted March 19, 1980; accepted for publication June 20, 1980.

 0571-7132/81/1701-0078$07.50