Self-similar expansion of finite-size non-quasi-neutral plasmas into vacuum: Relation to the problem of ion acceleration

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A new self-similar solution is presented which describes nonrelativistic expansion of a finite plasma mass into vacuum with a full account of charge separation effects. The solution exists only when the ratio $\Lambda = R/\lambda_D$ of the plasma scale length R to the Debye length λ_D is invariant, i.e., under the condition $T_e(t) \propto [n_e(t)]^{1-2/\nu}$, where $\nu=1$, 2, and 3 corresponds, respectively, to the planar, cylindrical, and spherical geometries. For $\Lambda \gg 1$ the position of the ion front and the maximum energy $\mathcal{E}_{i,\max}$ of accelerated ions are calculated analytically: in particular, for $\nu=3$ one finds $\mathcal{E}_{i,\max} = 2ZT_{e0}W(\Lambda^2/2)$, where T_{e0} is the initial electron temperature, Z is the ion charge, and W is the Lambert W function. It is argued that, when properly formulated, the results for $\mathcal{E}_{i,\max}$ can be applied more generally than the self-similar solution itself. Generalization to a two-temperature electron system reveals the conditions under which the high-energy tail of accelerated ions is determined solely by the hot-electron population.
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I. INTRODUCTION

Plasma expansion into vacuum has been a subject of considerable interest over the last 40 years in different branches of physics, and especially in relation to the interaction of intense laser pulses with matter.[1–8] In particular, the energy spectrum and maximum kinetic energy of accelerated ions were issues of controversy in many theoretical works.[2,4,8,9] Energetic ions, originating from the periphery of the expanding plasma, are crucially affected by the properties of the electron sheath extending into vacuum beyond the ion front. Nevertheless, most of the relevant analytical work was based on the quasineutral assumption.[1–4,10,11]

In this paper we present a rigorous self-similar solution to the system of two-fluid equations which describe the free expansion of a finite plasma mass. It is the first analytical solution which treats the effect of charge separation in a fully consistent way and allows a self-consistent determination of the position of the ion front and the maximum energy of accelerated ions $\mathcal{E}_{i,\max}$. The analysis of this solution leads to a conclusion that, besides the temperature T_e of the heated electrons, the value of $\mathcal{E}_{i,\max}$ is controlled by the plasma size expressed in terms of the Debye length. To the best of our knowledge, this fact has not been recognized in the previous publications on the subject.

The structure of the paper is as follows. In Sec. II we construct the self-similar solution. The energy spectrum of accelerated ions is analyzed in Sec. III. In Sec. IV we argue that the analytical formula for $\mathcal{E}_{i,\max}$, obtained from the rigorous solution valid for certain fixed values of the polytropic index γ, can in fact be extended to many practical situations with other values of γ. In Sec. V we address another problem of practical interest, namely, acceleration of light ions (in the test ion approximation) that may contaminate the surface of the main plasma sample.[12] In Sec. VI it is shown how our solution can be extended to a two-temperature electron system, and the conditions are formulated under which the two-temperature problem reduces to the single-temperature one.

II. SELF-SIMILAR SOLUTION

Suppose that at $t=0$ the electron component of a finite plasma sample is rapidly heated to a uniform temperature T_{e0} and that the subsequent plasma expansion is described by a nonrelativistic model of two charged fluids coupled via a self-consistent electric field. It is assumed that the ions remain cold, whereas the electron fluid obeys the thermodynamics of an ideal gas with a spatially uniform temperature distribution $T_e = T_e(t)$. The one-dimensional hydrodynamics of such a system is governed by the following equations:[4]

$$\frac{\partial n_{i(e)}}{\partial t} + \frac{1}{r^{\nu-1}}\frac{\partial}{\partial r}[r^{\nu-1} v_{i(e)} n_{i(e)}] = 0, \tag{1}$$

$$\frac{\partial v_i}{\partial t} + v_i \frac{\partial v_i}{\partial r} + \frac{Ze}{m_i}\frac{\partial \Phi}{\partial r} = 0, \tag{2}$$

$$\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial r} + \frac{T_e}{m_e n_e}\frac{\partial n_e}{\partial r} - \frac{e}{m_e}\frac{\partial \Phi}{\partial r} = 0, \tag{3}$$

$$\frac{1}{r^{\nu-1}}\frac{\partial}{\partial r}\left(r^{\nu-1}\frac{\partial \Phi}{\partial r}\right) = 4\pi e(n_e - Zn_i). \tag{4}$$

Here $\nu=1$, 2, and 3 correspond, respectively, to the planar, cylindrical, and spherical expansion geometries. Equation (1) combines two continuity equations for the electron (subscript e) and ion (subscript i) fluids. The Poisson equation (4) for the electrostatic potential Φ is written in cgs units, e is the elementary charge, and Z is the fixed ionization stage. Instead of solving the energy equation, we use a polytropic law

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$T_e(t)/T_{e0}=[n_e(t,0)/n_e(0,0)]^{\gamma-1}$ for the electron temperature evolution. For the moment, the polytropic index γ is supposed to be a free parameter. When γ coincides with the true adiabatic index of the expanding electron gas, the total plasma energy is conserved; otherwise, a corresponding external heating (or cooling) mechanism is assumed.[4] These are the usual assumptions whenever an analytical approach is attempted to the problem of free plasma expansion.[2,4,8]

A conventional wisdom in the theory of plasma expansion has been that the self-similar approach is possible only under the condition of quasineutrality $n_e=Zn_i$. Here we demonstrate that an important self-similar solution of Eqs. (1)–(4) can be constructed for a non-quasi-neutral plasma by using the well-known similarity ansatz[13]

$$v_{i(e)}(t,r)=\dot{R}\xi, \quad \xi=\frac{r}{R(t)}, \quad \dot{R}\equiv\frac{dR}{dt}, \tag{5}$$

$$n_e(t,r)=n_{e0}\left(\frac{R_0}{R}\right)^{\nu}N_e(\xi), \quad N_e(0)=1, \tag{6}$$

$$Zn_i(t,r)=n_{e0}\left(\frac{R_0}{R}\right)^{\nu}N_i(\xi), \quad N_i(0)\neq 1, \tag{7}$$

with a linear velocity profile. The principal rationale behind this representation is that a linear velocity-radius relation is a correct limit for the asymptotic stage of expansion of a finite fluid mass whenever its characteristic size $R(t)$ greatly exceeds the initial value $R_0=R(0)$.

Unlike in quasineutral hydrodynamics,[13] representations (5)–(7) generally do not lead to a self-similar solution of Eqs. (1)–(4) because the present system has two characteristic scale lengths, i.e., one is the plasma size $R(t)$ and the other is the Debye length $\lambda_D(t)=\sqrt{T_e/4\pi n_e e^2}$ brought in with the Poisson equation (4), where the characteristic electron density can be evaluated at the center $r=0$. However, this in turn means that if $R(t)$ and $\lambda_D(t)$ evolve coherently in time, in other words, if the ratio between them is kept constant as a single and invariant dimensionless parameter

$$\Lambda=\frac{R}{\lambda_D}=\frac{R_0}{\lambda_{D0}}=R_0\left(\frac{4\pi e^2 n_{e0}}{T_{e0}}\right)^{1/2}, \tag{8}$$

then one can expect to find a self-similar solution; this is indeed the case for the present system as shown below. Together with the mass conservation law, $n_e(t,0)\propto[R(t)]^{-\nu}$, Eq. (8) is reduced to

$$T_e R^{\nu-2}=T_{e0}R_0^{\nu-2}=\text{const}. \tag{9}$$

Comparing the polytropic law, $T_e(t)\propto[n_e(t,0)]^{\gamma-1}\propto[R(t)]^{-\nu(\gamma-1)}$, with Eq. (9), it is found that one can obtain a solution only in the special case of

$$\gamma=2-2/\nu. \tag{10}$$

At first glance, condition (10) may appear as rather artificial and restrictive. However, the values of γ prescribed by Eq. (10) can hardly be considered artificial for the cylindrical and spherical flow geometries. In the cylindrical case ($\nu=2,\gamma=1$) we are limited to the familiar isothermal approximation, which has always been one of the most widely used idealizations in theoretical studies on the subject. The spherical case with $\nu=3$, $\gamma=4/3$ turns out to be particularly interesting from the conceptual point of view because it allows an insight into what happens in the adiabatic case, when the total energy of the heated plasma mass is conserved (note that $\gamma=4/3$ is the true adiabatic index of the electron gas with $T_e\gg m_e c^2$). Apart from these considerations, below we present arguments why and how our new results, obtained for the specific case of $\gamma=2-2/\nu$, can be extended to other values of γ that might be more appropriate under the actual experimental conditions.

The next important point about the present solution is that the ion fluid is assumed to have a finite radial extension $0\leq r\leq R\xi_f$, i.e., a sharp edge at a fixed value ξ_f of the (Lagrangian) self-similar variable ξ. The presence of such a sharp edge is a natural consequence of the assumption of ions being cold and having a sharp external boundary at $t=0$.[8] The latter means that the functions N_i and $v_i/\dot{R}$ are defined only inside the interval $0\leq\xi\leq\xi_f$. The electron fluid, on the contrary, extends to infinity, and the functions $\mathbf{\Phi}$, N_e, and $v_e/\dot{R}$ are defined for all $0\leq\xi<\infty$. The region $\xi_f<\xi<\infty$, where $n_i=0$, comprises the electron sheath.

Equations (5)–(7) guarantee that the continuity equation (1) is automatically satisfied for any $R(t)$, $N_e(\xi)$, and $N_i(\xi)$. In the ion momentum equation (2), which should only be considered at $0\leq\xi\leq\xi_f$, variables separate to yield

$$\frac{m_i R}{ZT_e}\frac{d^2R}{dt^2}=\frac{R^{\nu-1}}{c_{s0}^2 R_0^{\nu-2}}\frac{d^2R}{dt^2}=-\frac{1}{\xi}\frac{d\phi}{d\xi}=2. \tag{11}$$

Here we used Eq. (9) to get rid of the temporal dependence $T_e(t)$, and introduced a dimensionless potential

$$\phi(\xi)=e\mathbf{\Phi}/T_e \tag{12}$$

with the boundary condition $\phi(0)=0$; $c_{s0}=(ZT_{e0}/m_i)^{1/2}$ is the sound speed at $t=0$. No generality is lost by choosing the separation constant equal to 2 because of the freedom in the normalization of R_0 and ξ. The first integral of the temporal part of Eq. (11) is

$$\dot{R}(t)=\begin{cases}2c_{s0}^2 t/R_0, & \nu=1,\\ 2c_{s0}\sqrt{\ln[R(t)/R_0]}, & \nu=2,\\ 2c_{s0}\sqrt{1-R_0/R(t)}, & \nu=3.\end{cases} \tag{13}$$

The electron momentum equation (3) applies to the entire domain $0\leq\xi<\infty$. With the aid of Eqs. (5) and (6) and the temporal part of (11), it can be integrated to yield

$$N_e=\exp(\phi-\mu_e\xi^2), \tag{14}$$

where $\mu_e=Zm_e/m_i\ll 1$ is the electron-to-ion mass-over-charge ratio. As might be expected,[4] for $\mu_e=0$ we recover the familiar Boltzmann relation, which is usually employed to close the system of the ion fluid equations [(1), (2), and (4)] without solving those for the electron fluid. The reason why we cannot stay within this conventional approach is as follows.

When applying the Boltzmann relation $n_e(t,r)=n_e(t,0)\exp(e\mathbf{\Phi}/T_e)$ to a dynamic problem, one actually assumes that, at any time t, electrons instantaneously relax to a thermodynamic equilibrium in a given electrostatic potential $\mathbf{\Phi}=\mathbf{\Phi}(t,r)$. However, similar to the case of the gravitational field,[14] such an equilibrium, being possible in the planar geometry, does not exist for spherically symmetric finite charge distributions. This is caused by the finite potential difference $\phi_\infty > -\infty$ between the plasma center and infinity (see Fig. 2 below), which, by virtue of $N_e(\infty)=\exp(\phi_\infty)>0$, implies that the global plasma neutrality can never be ensured with $\mu_e=0$. Consequently, if we choose $\mu_e=0$ (i.e., apply the usual Boltzmann relation), we can solve our problem for $\nu=1$, but not for $\nu=3$. The cylindrical case of $\nu=2$ is less obvious, but can be proven to fall in the same category as the spherical one. To overcome this difficulty, we resort to a fully dynamic treatment of the electron fluid with nonzero mass density ($\mu_e>0$), which leads us to Eq. (14).

It is worth noting that in the context of our problem—unlike, for example, in Ref. 15 where the ion fluid is treated as static—the mere fact of divergence of the potential $\mathbf{\Phi}(t,r)\to-\infty$ at $r\to\infty$, "enforced" by the Boltzmann relation, does not necessarily lead to an infinite energy of accelerated ions: because the electrostatic potential $\mathbf{\Phi}(t,r)$ depends on time, the electric field at the ion front may decay rapidly enough to result in a finite limiting velocity of accelerated ions at $t\to\infty$. In other words, the predicament with the Boltzmann relation is not due to the ensuing potential divergence: for our problem, it is geometry-dependent and caused by the absence of corresponding equilibrium solutions for cylindrical and spherical configurations.

The Poisson equation with respect to the self-similar variable ξ becomes

$$\frac{1}{\xi^{\nu-1}}\frac{d}{d\xi}\left(\xi^{\nu-1}\frac{d\phi}{d\xi}\right)=\begin{cases}\Lambda^2(N_e-N_i), & \xi\leqslant\xi_f,\\ \Lambda^2\exp(\phi-\mu_e\xi^2), & \xi>\xi_f,\end{cases} \tag{15}$$

where $\Lambda=R/\lambda_D$ is the key dimensionless parameter of our problem as defined in Eq. (8), and condition (9) ensures that Λ remains invariant during the expansion. Combining Eqs. (11), (14), and (15), we find all the profiles in the ion-fluid region $\xi\leqslant\xi_f$,

$$\phi=-\xi^2, \tag{16}$$

$$N_e=\exp[-(1+\mu_e)\xi^2], \tag{17}$$

$$N_i=N_e+2\nu\Lambda^{-2}. \tag{18}$$

Finally, to find the position of the ion front ξ_f, we have to solve Eq. (15) in the electron sheath region $\xi>\xi_f$ with the following boundary conditions:

$$\phi(\xi_f)=-\xi_f^2,\quad \frac{d\phi(\xi_f)}{d\xi}=-2\xi_f,\quad \lim_{\xi\to\infty}\xi^{\nu-1}\frac{d\phi}{d\xi}=0. \tag{19}$$

Three boundary conditions for the second-order equation (15) allow the unique determination of the function $\phi(\xi)$ and the parameter ξ_f. Generally, this eigenvalue problem must be solved numerically, and corresponding examples are shown in Figs. 1 and 2. An analytical solution can only be obtained in the planar case with $\mu_e=0$, and in the asymptotic limits of small and large values of Λ for any ν and $\mu_e\geqslant 0$.

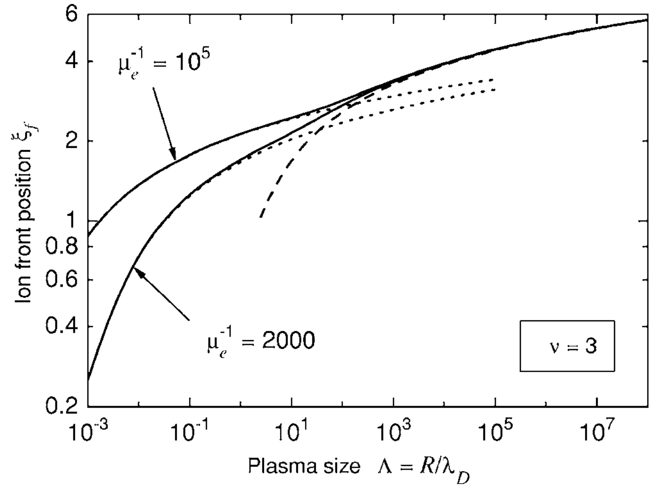

FIG. 1. Position of the ion front ξ_f as a function of the Λ parameter for $\nu=3$ and two values of the $\mu_e^{-1}=m_i/Zm_e$ parameter. The dashed and dotted curves represent, respectively, the asymptotic expressions (23) and (25).

In the simplest case of $\mu_e=0$ and $\nu=1$, integration of Eq. (15) for $\xi>\xi_f$ yields

$$N_e=\exp(\phi)=2\Lambda^{-2}(\xi-\xi_f+\xi_f^{-1})^{-2}, \tag{20}$$

$$\xi_f^2=W(\Lambda^2/2). \tag{21}$$

Here $W(x)$ is the inverse of the function

$$x=W\exp(W), \tag{22}$$

which is called the Lambert W function;[16] asymptotically, $W(x)\approx x$ for $x\ll 1$ and $W(x)\approx\ln(x/\ln x)$ for $x\gg 1$.

When $\Lambda\gg 1$ and $\mu_e\ll 1$, we recover essentially the same result (21) in all the three geometries $\nu=1$, 2, and 3. Indeed, the second condition in (19) tells us that in the vicinity of $\xi=\xi_f$, the electron density $N_e=\exp(\phi-\mu_e\xi^2)$ decays on a scale of $\Delta\xi\simeq\xi_f^{-1}\ll\xi_f$, provided that $\xi_f^2\gg 1$. In such a case one can neglect the curvature effects, represented by the factor $\xi^{\nu-1}$ in the divergence operator on the left-hand side of Eq. (15), and use $\mu_e\xi_f^2$ instead of $\mu_e\xi^2$ for $\mu_e\ll 1$. Then, by analogy with the $\nu=1$, $\mu_e=0$ case one obtains

$$\xi_f^2\approx(1+\mu_e)^{-1}W[(1+\mu_e)\Lambda^2/2], \tag{23}$$

which is in practice indistinguishable from Eq. (21).

In the opposite limit of $\Lambda\ll\mu_e^{1/2}$ for $\nu=1$ and 2, and $\Lambda\ll\mu_e^{3/4}$ for $\nu=3$ one calculates

$$\xi_f^2\approx\left[\frac{1}{4}\Gamma(\nu/2)\right]^{2/\nu}\frac{\Lambda^{4/\nu}}{\mu_e} \tag{24}$$

by setting $\phi=0$ at $\xi>\xi_f$ and integrating Eq. (15) from $\xi=0$ to $\xi=\infty$; here $\Gamma(x)$ is the gamma function. In the most interesting spherical case the low-Λ limit (24) can be significantly improved by using $\phi=-3\xi_f^2$ (the potential at infinity when the electron cloud at $\xi>\xi_f$ is ignored) instead of $\phi=0$ at $\xi>\xi_f$ to get

$$\xi_f^2\approx\frac{1}{2}W\left(\frac{\pi^{1/3}}{2\mu_e}\Lambda^{4/3}\right). \tag{25}$$

Figure 1 shows the values of $\xi_f(\Lambda)$ as calculated numeri-

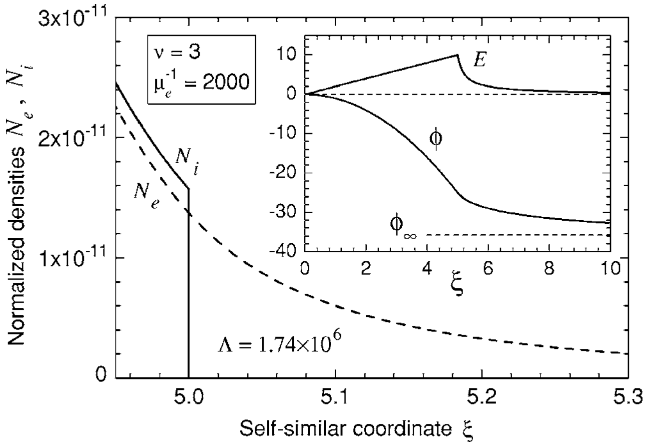

FIG. 2. Spatial profiles for the normalized ion N_i and electron N_e densities near the ion front ($\xi=\xi_f=5$) together with the potential ϕ and electric field $E=-d\phi/d\xi$ as calculated for $\nu=3$ and $\mu_e^{-1}=2000$.

cally for the case of spherical expansion with two values of the $m_i/Zm_e=\mu_e^{-1}$ ratio, roughly spanning the range of its possible variation. It is seen that the two asymptotic formulas [(23) and (25)] quite accurately cover the entire range of Λ values. Note that formally $\xi_f\to\infty$ as $\Lambda\to\infty$, although the values of $\xi_f\gtrsim 6$ ($\Lambda\gtrsim 10^9$) can hardly be of any practical significance. Spatial profiles of the normalized densities, N_i and N_e, the potential ϕ, and the electric field $E=-d\phi/d\xi$ are shown in Fig. 2 for the case of $\nu=3$ and $\mu_e^{-1}=2000$. Here, to ease the perception, the ion front position is chosen to be at $\xi_f=5$, which corresponds to $\Lambda=1.74\times10^6$. One clearly observes a steep drop of N_e and E at $\xi>\xi_f$ on a scale of $\Delta\xi\ll\xi_f$—an approximation needed to derive Eq. (23).

By its physical meaning, the limit of $\Lambda\to\infty$ corresponds to the quasineutral hydrodynamics, and one readily verifies that our solution in this limit does indeed approach the corresponding self-similar solution of the usual fluid dynamics. In the opposite case of $\Lambda\ll 1$, when the ion density $N_i\approx 2\nu\Lambda^{-2}\gg 1$ greatly exceeds the electron density $N_e\leqslant 1$, our solution describes the Coulomb explosion of a bare ion sphere (or a cylinder, or a slab) that has suddenly been deprived of all its electrons. A remarkable fact is that, as $\Lambda\to 0$, the assumed linear velocity-radius relation is not only asymptotic for $t\to\infty$, but becomes exact for all $t\geqslant 0$ in the most common case of a uniform initial density profile.

III. ENERGY SPECTRUM OF ACCELERATED IONS

In general, the simplest way to evaluate the ion energy spectrum by free plasma expansion would be to ignore the effects of charge separation and solve the appropriate hydrodynamical problem in the quasineutral approximation. In our case, when a uniform distribution of the electron temperature is assumed, a corresponding quasineutral self-similar solution is easily obtained by assuming $n_e=Zn_i$ in Eqs. (1)–(7). It has a Gaussian density profile, $N_e(\xi)=N_i(\xi)=\exp(-\xi^2)$, which extends to infinity. The asymptotic (at $t\to\infty$) velocity distribution is $v_{i(e)}(t,\xi)=v_\infty\xi$, where

$$v_\infty=\lim_{t\to\infty}\dot{R}=\begin{cases}2c_{s0}\sqrt{\ln(R/R_0)}, & \gamma=1,\\ 2c_{s0}/\sqrt{\nu(\gamma-1)}, & \gamma>1,\end{cases} \tag{26}$$

may be called the bulk expansion velocity. Clearly, a substantial portion of the plasma ions will have kinetic energies around the value

$$\mathcal{E}_0=\frac{1}{2}m_iv_\infty^2 \tag{27}$$

($\mathcal{E}_0=2ZT_{e0}$ in the particular case of $\nu=3$, $\gamma=4/3$ described by the self-similar solution of Sec. II). The bulk ion energy $\mathcal{E}_0$ is finite for any $\gamma>1$, but diverges logarithmically with t in the isothermal case $\gamma=1$, which requires continuous plasma heating from the external energy reservoir. Note that this latter divergence is of pure hydrodynamical origin and has nothing to do with the possible divergence of the electrostatic potential at $r\to\infty$.

Now, if one asks by what factor the maximum energy $\mathcal{E}_{i,\max}$ of accelerated ions may exceed $\mathcal{E}_0$, the answer given by the quasineutral solution will be *infinite*, i.e., there always exists an exponentially small fraction of ions accelerated to arbitrarily high velocities—even when $\gamma>1$ and the expanding plasma has a finite amount of the total energy. A *finite* value of the ratio $\mathcal{E}_{i,\max}/\mathcal{E}_0$ is obtained only after one takes into account the effect of charge separation and solves the problem in the two-fluid approximation. Our self-similar solution gives a simple expression

$$\mathcal{E}_{i,\max}=\mathcal{E}_0\xi_f^2, \tag{28}$$

where ξ_f is found by solving the eigenvalue problems [(15) and (19)]. Note that ξ_f remains finite even when, to derive Eq. (23), the electron sheath is treated in the planar approximation, which implies a formal divergence of the potential $\boldsymbol{\Phi}(t,r)$ at $r\to\infty$.

Our most significant new result is that the key parameter, which determines the value of the plasma acceleration factor ξ_f^2, is the plasma size Λ in units of the Debye length, and the value of ξ_f^2 is given by a simple formula (21). The approximate formula (21) applies when $\mu_e\ll 1$ and Λ is sufficiently large: $\Lambda\gtrsim\Lambda_*=0.1$, 5, and 50 for $\nu=1$, 2, and 3, respectively; otherwise, the next important parameter, $\mu_e=Zm_e/m_i$, comes into play. As $\Lambda\to\infty$, the acceleration factor ξ_f^2 becomes infinite in agreement with the quasineutral solution.

In view of Eq. (5), the distribution of accelerated ions over their energy $\mathcal{E}_i=\mathcal{E}_0\xi^2$ at $t\to\infty$ is a simple image

$$\frac{d\mathcal{N}_i}{d\mathcal{E}_i}=\frac{A}{\mathcal{E}_0}\left(\frac{\mathcal{E}_i}{\mathcal{E}_0}\right)^{\nu/2-1}\left\{\frac{2\nu}{\Lambda^2}+\exp\left[-(1+\mu_e)\frac{\mathcal{E}_i}{\mathcal{E}_0}\right]\right\} \tag{29}$$

of the spatial distribution $N_i(\xi)$, which is confined to a finite-energy interval $0\leqslant\mathcal{E}_i\leqslant\mathcal{E}_{i,\max}$; when the total number of ions $\mathcal{N}_i$ is normalized to unity, the normalization constant is $A=(2\int_0^{\xi_f}N_i\xi^{\nu-1}d\xi)^{-1}$. Note that Eq. (23) in Ref. 17, for example, which is obtained under the quasineutral assumption, is found to be just an approximate form of the exact expression (29).

Of considerable interest in some applications is the total number of ions accelerated to maximum energies $\mathcal{E}_i \simeq \mathcal{E}_{i,\max}$. From Eqs. (21), (28), and (29) one obtains the following estimate for the fraction of such ions:

$$\Delta \mathcal{N}_{i,f} \approx \mathcal{E}_{i,\max} \left. \frac{d\mathcal{N}_i}{d\mathcal{E}_i} \right|_{\mathcal{E}_i=\mathcal{E}_{i,\max}} = \frac{2}{\Gamma(\nu/2)} \frac{\xi_f^{\nu+2}}{\Lambda^2}, \tag{30}$$

which is valid for $\Lambda \gg 1$ where Eq. (21) applies.

IV. APPLICATION TO PRACTICAL SITUATIONS

Our self-similar solution and the analytical expression for the maximum ion energy $\mathcal{E}_{i,\max}$ have been obtained for a specific value $\gamma=2-2/\nu$ of the polytropic index γ. Here we argue that in the cylindrical ($\nu=2$) and spherical ($\nu=3$) geometries this result can be easily extended to other values of $\gamma \geq 1$.

First of all, note that a weak (weaker than logarithmic) dependence of the acceleration factor ξ_f^2 in Eq. (21) on the Λ parameter suggests that the error introduced by the assumption R/λ_D=const [Eq. (9)] in situations with $\gamma \neq 2-2/\nu$ should be rather small for $\Lambda \gg 1$: when, for example, $\Lambda \geq 10^4$, a factor of 2 variation in the Λ value would imply less than a 10% variation of ξ_f^2. Also note that in one of the most interesting cases of $\nu=3$, $\gamma=5/3$, the Λ parameter in turn is a rather weak function of the plasma density, $\Lambda \propto n_e^{-1/6}$. Hence, for most practical purposes it should suffice to use the in-flight value of the Λ parameter measured at the appropriate phase of expansion where most of the ion acceleration occurs.

The latter can be identified by invoking the known expression of $E_f=(8\pi n_{ef} T_e)^{1/2} \propto n_{ef}^{\gamma/2}$ (Ref. 2) for the electric field at the ion front $r=r_f$, which, when combined with $n_{ef} \propto r_f^{-\nu}$, tells us that the maximum ion energy $\mathcal{E}_{i,\max} = Ze\int_{r_{f0}}^{\infty} E_f dr_f \propto \int_{r_{f0}}^{\infty} r_f^{-\gamma\nu/2} dr_f$ is determined by an integral that converges for $\gamma\nu > 2$. Convergence means that practically all the acceleration occurs during the initial phase of the expansion—provided, of course, that the asymptotic density and velocity profiles have already been established. Hence, our recipe for isothermal and adiabatic (or a combination of an isothermal stage followed by the adiabatic one, as considered in Ref. 17) expansions of cylindrical and spherical plasmas is as follows: Calculate the quasineutral hydrodynamics of expansion and evaluate the Λ parameter when the central density drops by, say, a factor $\simeq 2$; then (provided that $\Lambda > \Lambda_*$), calculate the maximum ion energy from Eqs. (23) and (28). A special case of $\nu=2$, $\gamma=1$ with $\gamma\nu=2$ is described explicitly by our solution. In the planar case, finite values of $\mathcal{E}_{i,\max}$ can be expected only for $\gamma > 2$. A proper verification of our prescription for $\gamma \neq 2-2/\nu$ requires numerical simulations of the same sort as in Ref. 8 and goes beyond the scope of this work.

Direct application of our results formally requires the hydrodynamic expansion to occur slower than (i) the electron-electron (e-e) collisions can establish the Maxwellian distribution ($t_{ee} \ll R_0/c_{s0}$) and (ii) the electron heat conduction can effectively level off the electron temperature across the heated sample ($t_{\text{ec}} \simeq n_{e0} R_0^2/\kappa_e \ll R_0/c_{s0}$). For the adiabatic expansion, the appropriate value of γ depends on the number of degrees of freedom over which the e-e relaxation takes place. As a typical example, for a few-micrometer-thick carbon sample with the e-e relaxation over the three degrees of freedom ($\gamma=5/3$ when T_e is nonrelativistic), the above conditions will be fulfilled in the temperature interval

$$0.4(n_{e0}R_0)_{20}^{1/2}\ \text{keV} \ll T_{e0} \ll 50(n_{e0}R_0)_{20}^{1/2}\ \text{keV}, \tag{31}$$

where $(n_{e0}R_0)_{20}$ is the value of $n_{e0}R_0$ in units of 10^{20} cm^{-2}; for the electron heat conductivity κ_e we used the Spitzer value with the Coulomb logarithm equal to 5. Note that, by setting $\nu=1$ and $\gamma=3$, our result for the maximum ion energy can also be applied to the adiabatic expansion of a finite planar plasma with a one-dimensional velocity distribution of hot electrons treated in Ref. 10 in the quasineutral approximation.

V. TEST ION ACCELERATION

So far, we have considered only a single-species ion problem. Now that we know the self-consistent electric field of such a single-species plasma, we can apply it to another problem of practical interest, namely, to the acceleration of a test ion with different values of A and Z that might be initially present on the target surface.[12] A typical situation that one may keep in mind would, for example, be contaminating hydrogen atoms at the surface of a high-Z sample.

Suppose that a test ion with a mass m_p and an electric charge $+Z_p e$ is initially put on the surface of a bulk ion sphere ($\nu=3$) described by our self-similar solution. The nonrelativistic dynamics of such a test ion will be governed by the following equations:

$$\frac{dr_p}{dt} = v_p, \tag{32}$$

$$\frac{dv_p}{dt} = -\frac{Z_p e}{m_p} \frac{\partial \Phi(t, r_p)}{\partial r}, \tag{33}$$

where $r_p(t)$ and $v_p(t)$ denote the position and velocity of the test ion, respectively. Equation (13) enables us to switch from the independent variable t to R and, by using Eqs. (5) and (12), transform Eqs. (32) and (33) into

$$\frac{dr_p}{dR} = \frac{v_p}{2c_{s0}\sqrt{1-R_0/R}}, \tag{34}$$

$$\frac{dv_p}{dR} = -\frac{\alpha c_{s0} R_0}{2R^2\sqrt{1-R_0/R}} \frac{d\phi(\xi_p)}{d\xi}, \tag{35}$$

where $\xi_p(R)=r_p(R)/R$, and

$$\alpha = \frac{Z_p m_i}{m_p Z} \tag{36}$$

is a new dimensionless parameter which characterizes the charge-over-mass ratio of the test ion.

Here we are interested in the asymptotic velocity ratio $v_{p\infty}/v_{f\infty} = \xi_p(\infty)/\xi_f$ between the test ion and the front of the background ions in the limit of $t \to \infty$. This ratio is a function of only four dimensionless parameters ν, μ_e, Λ, and α. Fig-

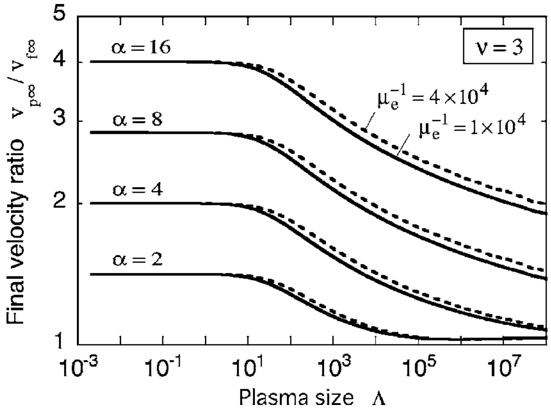

FIG. 3. Final velocity ratio between the test ion and the background ion front for different values of the normalized charge-over-mass ratio α defined by Eq. (36).

ure 3 shows the dependence of the $v_{p\infty}/v_{f\infty}$ ratio on Λ as calculated numerically in the case of $\nu=3$ for four different values of α and two values of μ_e. It is seen that a test ion is accelerated to a relatively higher final velocity for larger values of α and smaller values of Λ. In particular, as one approaches the conditions of the Coulomb explosion with $\Lambda \ll 1$, the ratio $v_{p\infty}/v_{f\infty}$ approaches the value $v_{p\infty}/v_{f\infty}=\sqrt{\alpha}$. The latter can be easily understood after one recalls that in such a situation a test ion with $\alpha \geqslant 1$ "feels" a fixed amount of positive charge behind itself all the time.

VI. TWO-TEMPERATURE ELECTRON SYSTEM

A fruitful model for interpreting many laser-plasma experiments has been that of two interpenetrating electron populations with differing temperatures.[18–22] Our solution from Sec. II admits a straightforward generalization to a multi-electron-fluid case under the condition that all the electron populations obey a polytropic law $T_e \propto n_e^\gamma$ with the same polytropic index (10). Here we describe the two-temperature case, with $T_h=T_h(t)$ and $T_c=T_c(t)$ being, respectively, the temperatures of the hot and cold electrons. Their fixed ratio

$$\theta_h = \frac{T_h(t)}{T_c(t)} = \text{const} > 1 \tag{37}$$

is a free parameter of the model.

For each of the two electron fluids we have separate continuity and motion equations that are fully analogous to Eqs. (1) and (3); we denote the densities [velocities] of the hot and cold electrons as $n_{\text{eh}}(t,r)$ $[v_{\text{eh}}(t,r)]$ and $n_{\text{ec}}(t,r)$ $[v_{\text{ec}}(t,r)]$, respectively. In the similarity ansatz, we choose to normalize the densities and temperatures with respect to the hot-electron values

$$n_{\text{eh}0} = n_{\text{eh}}(0,0), \quad T_{h0} = T_h(0). \tag{38}$$

The reason for this choice will become clear below.

In the self-similar representation, all the three fluids have equal velocities given by Eq. (5); the electron densities are cast in the following form:

$$n_{\text{eh}}(t,r) = n_{\text{eh}0}\left(\frac{R_0}{R}\right)^\nu N_{\text{eh}}(\xi), \quad N_{\text{eh}}(0) = 1, \tag{39}$$

$$n_{\text{ec}}(t,r) = n_{\text{eh}0}\left(\frac{R_0}{R}\right)^\nu N_{\text{ec}}(\xi), \quad N_{\text{ec}}(0) = \chi_c, \tag{40}$$

where χ_c is another free parameter of the model, which can be related to the total fraction of the hot electrons [see Eq. (52) below]; the ion density is given by the old Eq. (7) with $n_{e0}=n_{\text{eh}0}$.

Once we normalize the electrostatic potential $\mathbf{\Phi}$ to the hot-electron temperature,

$$\phi(\xi) = \frac{e\mathbf{\Phi}}{T_h}, \quad \phi(0) = 0, \tag{41}$$

we obtain the same equations [(11) and (13)] of ion motion with T_e replaced by T_h and $c_{s0}=(ZT_{h0}/m_i)^{1/2}$. The electron momentum equations yield

$$N_{\text{eh}} = \exp\left(\phi - \mu_e\xi^2\right), \tag{42}$$

$$N_{\text{ec}} = \chi_c \exp[\theta_h(\phi - \mu_e\xi^2)], \tag{43}$$

for the entire range of $0 \leqslant \xi < \infty$. As before, in the ion fluid region $0 \leqslant \xi \leqslant \xi_f$ we have

$$\phi = -\xi^2, \tag{44}$$

$$N_{\text{eh}} = \exp[-(1+\mu_e)\xi^2], \tag{45}$$

$$N_{\text{ec}} = \chi_c \exp[-\theta_h(1+\mu_e)\xi^2], \tag{46}$$

$$N_i = N_{\text{eh}} + N_{\text{ec}} + 2\nu\Lambda_h^{-2}. \tag{47}$$

To find the position of the ion front ξ_f, we have to solve the Poisson equation

$$\frac{1}{\xi^{\nu-1}}\frac{d}{d\xi}\left(\xi^{\nu-1}\frac{d\phi}{d\xi}\right) = \Lambda_h^2\{\exp(\phi - \mu_e\xi^2) + \chi_c \exp[\theta_h(\phi - \mu_e\xi^2)]\} \tag{48}$$

in the region $\xi_f \leqslant \xi < \infty$ with the boundary conditions (19). The parameter

$$\Lambda_h = R_0\left(\frac{4\pi e^2 n_{\text{eh}0}}{T_{h0}}\right)^{1/2} \tag{49}$$

is the size of the hot-electron system in units of its own Debye length.

It is easy to see that for temperature ratios in excess of $\theta_h=10-20$, which appear to be of main interest for applications,[19,21,22] the second term on the right-hand side of Eq. (48) is typically much smaller than the first one. The latter means that the contribution of cold electrons can be ignored altogether when calculating the position of the ion front and the maximum ion energy, i.e., the two-temperature problem essentially reduces to the single-temperature one. A more accurate condition for this can be derived as follows.

If we consider the case of $\xi_f^2 \gg 1$, we can, similar to the logic behind Eq. (23), neglect the curvature effects in the divergence operator in Eq. (48) and, having performed a single integration and used the boundary conditions (19), derive the following equation for ξ_f^2:

$$\frac{\xi_f^2 \exp(\xi_f^2)}{1+\chi_c\theta_h^{-1}\exp[-(\theta_h-1)\xi_f^2]} = \frac{\Lambda_h^2}{2} \tag{50}$$

(here we have ignored also the small quantity μ_e). Clearly, once

$$\xi_f^2 > \frac{1}{\theta_h - 1}\ln\frac{\chi_c}{\theta_h}, \tag{51}$$

the second term in the denominator on the left-hand side of Eq. (50) can be neglected, and we recover our old formula (21). Inequality (51) is a condition under which the two-temperature problem reduces to that of a single population of hot electrons. It is actually applicable whenever $\Lambda_h \gtrsim 1$. For $\theta_h \geqslant 20$ condition (51) will be fulfilled for any conceivable value of the density ratio χ_c that might be realized in experiments. Its physical content is rather transparent: the pressure of the cold electrons at the ion front $\xi=\xi_f$ must be smaller than that of the hot electrons.

From a practical point of view, it may be useful to relate the parameters χ_c and θ_h to the total fraction of hot electrons $\mathcal{N}_{\rm eh}/\mathcal{N}_{\rm ec}$. Again, a simple analytical estimate can be obtained in the limit of $\xi_f^2 \gg 1$ by integrating Eqs. (45) and (46) over the entire range of $0 \leqslant \xi < \infty$:

$$\frac{\mathcal{N}_{\rm eh}}{\mathcal{N}_{\rm ec}} \approx \theta_h^{\nu/2}\chi_c^{-1}. \tag{52}$$

One readily sees that for $\theta_h \gtrsim 20$ condition (51) may be satisfied even when $\mathcal{N}_{\rm eh} \ll \mathcal{N}_{\rm ec}$. From this we conclude that, once the temperature ratio between the hot and cold electrons is sufficiently high, the relative number of hot electrons becomes unimportant: the high-energy tail of accelerated ions is determined solely by the hot-electron population. It is only in the case of $\mathcal{N}_{\rm eh} \ll \mathcal{N}_{\rm ec}$ that our estimate (30) for the fraction of fast ions must be multiplied by the small ratio (52).

When a two-electron population is treated in the quasineutral approximation, rarefaction shocks become possible for $\theta_h > 5+\sqrt{24}$.[19] In our case, where charge separation is treated in a consistent way, such shocks would appear as a continuous variation of the potential and electron density on a spatial scale of a local Debye length, i.e., on a scale of $\Delta\xi \simeq \Lambda_h^{-1} N_{\rm eh}^{-1/2}$ (or shorter). Apart from the outer ion edge, our self-similar solution contains no such structures because it represents the asymptotic stage of expansion of a finite plasma mass, when the original rarefaction wave has long since been reflected from the center. In this limit, as discussed in Ref. 20, the solution exhibits only two scale lengths (for $\theta_h \gg 1$ and $\chi_c \gg 1$), namely, $\Delta\xi \simeq 1$ in the hot-electron corona and $\Delta\xi \simeq \theta_h^{-1/2}$ in the cold-electron core [cf. Eqs. (45) and (46)].

VII. CONCLUSION

For the first time a self-similar solution has been constructed which accounts in a consistent way for the effect of charge separation on the ion acceleration by free expansion of a finite plasma mass into vacuum. This new solution clearly demonstrates that one of the key parameters which determines the final velocity of the expanding ion front is the initial plasma size Λ in units of the Debye length $\lambda_D = \sqrt{T_e/4\pi n_e e^2}$. The dependence of the maximum ion energy $\mathcal{E}_{i,\max}$ on Λ is weak and is given by a simple analytical formula (21) for $\Lambda \gg 1$. In contrast to the well-studied case of the isothermal expansion of a semi-infinite plasma wall,[8] a finite limiting velocity of the ion front is obtained in the cylindrical and spherical flow geometries for $\gamma > 1$, where γ is the effective polytropic index of the electron gas.

Mathematically, the present self-similar solution exists only for certain specific values of the polytropic index γ depending on the flow geometry. It is argued, however, that the underlying physics of the two-fluid plasma model and the form of the calculated dependences allow a straightforward generalization of the obtained results to other practical values of γ. The problem of test ion acceleration is also easily solved in the context of the present self-similar solution. The resulting enhancement factor $v_{p\infty}/v_{f\infty}$ for the final velocity of lighter test ions relative to the main ion front is defined primarily by the values of the Λ parameter and the normalized charge-over-mass ratio α of the test ion. Finally, the obtained solution is extended to a two-temperature electron distribution. It is shown that for sufficiently large temperature ratios between the hot and cold electrons, $T_h/T_c \gtrsim$ 10–20, the high-energy tail of accelerated ions is determined solely by the hot-electron population.

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