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THE METHOD OF ARTIFICIAL VISCOSITY FOR COMPUTING ONE-DIMENSIONAL FLOWS*

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A generalization of the artificial tensor viscosity, which was used earlier for spherical flows only, is proposed for each of the following three types of one-dimensional flows: flows in a plane, cylindrical flows, and spherical flows. The choice of specific values for the six free parameters of artificial viscosity of fairly general form considered in this paper is illustrated by examples of two types of flows, which can often be found in applied problems.

The approach based on introducing artificial viscosity (pseudoviscosity) is one of the most widely used methods for constructing numerical models of shock wave flows in ideal hydrodynamics /1/. The quadratic scalar pseudoviscosity (quadratic, that is, with respect to the velocity) which was proposed by Neumann and Richtmyer /2/, became a classical idea. Although the Neumann-Richtmyer pseudoviscosity provides a good description of the transmission of single plane shock waves, it has a number of drawbacks /3/. Parasitic warming (generation of entropy) under adiabatic contraction (in particular under homological contraction) and strong entropy perturbations occurring as the shock wave is reflected by a solid wall or by a centre of symmetry ("warming of the wall") are the most serious of the drawbacks from the point of view of applications. It is clear that no modification of the pseudoviscous terms will make it possible to eliminate the parasitic effects completely, but one can try to neutralize (or at least to reduce) the effects in hydrodynamical flows of specific types, which are important from the point of view of applications. With this end in view, in this paper we consider a fairly general form of artificial viscosity, which is represented as a linear combination of scalar and tensor components in conformity with the class of one-dimensional flows with a stationary centre. In the general case, a stationary centre is present for spherically symmetric and cylindrically symmetric flows only, while in plane geometry a subset is thereby selected in the set of all possible one-dimensional flows.

1. The equations of ideal hydrodynamics with artificial viscosity.

To start with we will consider the general form of the equation of motion for a viscous fluid /4/:

$$\rho\frac{\partial u_i}{\partial t}+\rho u_k\frac{\partial u_i}{\partial x_k}+\frac{\partial P}{\partial x_i}=\frac{\partial \sigma_{ik}}{\partial x_k}, \tag{1.1}$$

where

$$\sigma_{ik}=\zeta_s\delta_{ik}\frac{\partial u_j}{\partial x_j}+\zeta_t\left(\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}-\frac{2}{3}\delta_{ik}\frac{\partial u_j}{\partial x_j}\right) \tag{1.2}$$

is the tensor of viscous stresses (it is understood that summation is carried out for each pair of indices). The first term with the coefficient ζ_s on the right-hand side of (1.2) is the scalar viscosity, which can be represented as a simple addition to the pressure P. The second term with the coefficient ζ_t is the tensor viscosity with zero trace. The equation for the energy of a viscous fluid has the form

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$$\rho\frac{\partial \varepsilon}{\partial t}+\rho u_k\frac{\partial \varepsilon}{\partial x_k}+P\frac{\partial u_j}{\partial x_j}=\sigma_{ik}\frac{\partial u_i}{\partial x_k} \tag{1.3}$$

In the case of spherically symmetric flows, Eqs.(1.1)-(1.3) take the form

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial r}+\frac{\partial}{\partial r}\left[P-\zeta_*\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u)\right]= \tag{1.4}$$
$$\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{4}{3}\zeta_i\Sigma\right),$$

$$\rho\frac{\partial \varepsilon}{\partial t}+\rho u\frac{\partial \varepsilon}{\partial r}+\left[P-\zeta_*\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u)\right]\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u)=\frac{4}{3}\zeta_i\Sigma^2, \tag{1.5}$$

where

$$\Sigma=\frac{\partial u}{\partial r}-\frac{u}{r}=r\frac{\partial}{\partial r}\left(\frac{u}{r}\right). \tag{1.6}$$

From these equations one can see at once that when there is no scalar viscosity and the velocity $u(t, r)=C(t)r$, the tensor viscosity vanishes under homological spherical contraction and expansion. This means that by passing from the Neumann-Richtmyer scalar pseudoviscosity to the tensor pseudoviscosity (1.2) (see /5/), is is possible to eliminate the parasitic generation of entropy under homological spherically symmetric contraction. This property is of considerable importance for a number of applied problems.

It is well-known, however, that even in the case of homological cylindrically symmetric contraction, the tensor component of the physical viscosity (the first viscosity) does not vanish and, all the more, it does not vanish for homological parallel plane motion. Nevertheless, one would like to construct a pseudoviscosity that vanishes identically for homological one-dimensional flows in each of the three geometries. With this end in view, in the present paper we propose the following generalization of Eqs.(1.4) and (1.5) (combined with the equation of mass balance) to one-dimensional flows in an arbitrary geometry:

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu u\rho)=0, \tag{1.7}$$

$$\rho\frac{\partial u}{\partial t}+\rho u\frac{\partial u}{\partial r}+\frac{\partial}{\partial r}\left[P-\eta_*\frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu u)\right]= \tag{1.8}$$
$$\frac{1}{r^{\nu+1}}\frac{\partial}{\partial r}(r^{\nu+1}\eta_i\Sigma),$$

$$\rho\frac{\partial \varepsilon}{\partial t}+\rho u\frac{\partial \varepsilon}{\partial r}+\left[P-\eta_*\frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu u)\right]\frac{1}{r^\nu}\frac{\partial}{\partial r}(r^\nu u)=\eta_i\Sigma^2; \tag{1.9}$$

the value of Σ is given by (1.6).

In the spherical case ($\nu=2$) Eqs.(1.8) and (1.9) differ from the "physical" equations (1.1)-(1.3) only by the normalization of the coefficient of tensor viscosity, $\eta_i=4\zeta_i/3$. However, to get (1.8) and (1.9) from the physical Eqs.(1.1)-(1.3) in cylindrical geometry ($\nu=1$), it is necessary to make the substitutions $\zeta_i=\eta_i$, $\zeta_*=-\eta_i/3+\eta_*$. This means that in the cylindrical case Eqs.(1.8) and (1.9) are physically correct under the condition $\eta_*\geqslant\eta_i/3$. In other words, if one assumes that there is a fluid whose cylindrical motions are described by Eqs.(1.7)-(1.9) with $\nu=1$, $\eta_*\doteq0$, and $\eta_i>0$, then there are non-cylindrical motions of the fluid in a wider class of flows such that the viscous dissipation of mechanical energy is accompanied by a decrease in entropy. Generally speaking, in plane geometry ($\nu=0$) it is impossible to obtain (1.8) or (1.9) from the physical Eqs.(1.1)-(1.3).

Nevertheless, it is perfectly clear that if the three types of one-dimensional motions, namely the spherical, cylindrical, and flat motions, are regarded as being completely isolated from each other and from other wider classes of flows, then Eqs.(1.7)-(1.9) are completely correct for any $\eta_*\geqslant0$ and $\eta_i\geqslant0$ from the point of view of thermodynamics. In this case the viscosity with the coefficient η_i introduced on the right-hand sides of Eqs.(1.8) and (1.9) may no longer be called the tensor viscosity in the strict sense of the word. Therefore, to describe this viscosity, we will introduce the term t-viscosity. The basic property of the t-viscosity is that it vanishes identically for homological flows with $u=C(t)r$. In plane geometry the use of the artificial t-viscosity is justified only for the class of flows with a stationary centre $r=0$. The choice of the coefficients of pseudoviscosity η_* and η_i is discussed in the next section.

2. The finite-difference approximation.

The system of Eqs.(1.7)-(1.9) was solved numerically using the explicit difference scheme

$$\frac{\bar{r}-r}{\Delta t}=\frac{1}{2}(u+\bar{u}), \tag{2.1}$$

$$\frac{\bar{u}-u}{\Delta t}+2\bar{r}^{\nu}\frac{P'-P_{-}'}{\Delta m+\Delta m_{-}}=\frac{2}{\bar{r}}\frac{\bar{\eta}_t\Sigma-\bar{\eta}_{t-}\Sigma_{-}}{\Delta m+\Delta m_{-}}, \tag{2.2}$$

$$\varepsilon_T\frac{\bar{T}-T}{\Delta t}+(P'+\varepsilon_V)\dot{V}=\bar{\eta}_t\Sigma^2 \tag{2.3}$$

with respect to the Lagrange mass coordinate

$$m=\int_0^r \rho r^{\nu}\,dr.$$

r, u, and T are the basic dependent variables. The equation of state is assumed to be given in the form $P=P(V, T)$, $\varepsilon=\varepsilon(V, T)$, $\varepsilon_T=\partial\varepsilon/\partial T$, $\varepsilon_V=\partial\varepsilon/\partial V$, $V=1/\rho$. It is also assumed that all the quantities in Eqs.(2.1)-(2.3) have a time index n and a spatial index j. The notation

$$f_j^n=f,\quad f_j^{n+1}=\bar{f},\quad f_j^{n+1/2}=\tilde{f},\quad f_{j\pm1}^{n}=f_{\pm}$$

is used for conciseness. The quantities m_j, r_j and u_j are centred at the left-hand end of the j-th Lagrange interval, and $m_1=r_1=u_1=0$. The quantities T, V, $\dot{V}$, P, P', $\bar{\eta}_t$, $\bar{\eta}_{t'}$ and Σ are centred at the mid-point of the j-th interval. Eqs.(2.1)-(2.3) are supplemented by the relations

$$\tilde{r}=r+u\Delta t/2, \tag{2.4}$$

$$P'=P-\bar{\eta}_t\dot{V}, \tag{2.5}$$

$$\Sigma=\frac{1}{2\Delta m}\left(\frac{u_{+}+\bar{u}_{+}}{\bar{r}_{+}}-\frac{u+\bar{u}}{\bar{r}}\right), \tag{2.6}$$

$$\dot{V}=\frac{1}{2\Delta m}[(u_{+}+\bar{u}_{+})\bar{r}_{+}^{\nu}-(u+\bar{u})\bar{r}^{\nu}]. \tag{2.7}$$

We remark that in passing from (1.8) and (1.9) to (2.2) and (2.3), we changed the notation for the quantities η_t, $\eta_{t'}$, and Σ in an obvious way. Relation (2.2) represents a system of linear algebraic equations with respect to $\bar{u}_j$, which is easy to solve by the method of reduction. The density (the specific volume) is not one of the basic dependent variables and can be computed from the condition that ensures that the mass is preserved in each Lagrange interval:

$$\Delta m=m_{+}-m=\frac{\rho}{\nu+1}(\bar{r}_{+}^{\nu+1}-\bar{r}^{\nu+1}).$$

Conservation of total energy for $\varepsilon_T=\text{const}$ and $\varepsilon_V=0$ is an important property of the difference scheme (2.1)-(2.3). Indeed, according to (2.2) and (2.3), if the total energy of a Lagrange interval Δm is defined as

$$\Delta E=\varepsilon\Delta m+\begin{cases} {}^1/_8(\Delta m_1+\Delta m_2)u_2^2, & j=1,\\ {}^1/_8[(\Delta m_{-}+\Delta m)u^2+(\Delta m+\Delta m_{+})u_{+}^2], & j=2,3,\ldots,N-1,\\ {}^1/_8(\Delta m_{N-1}+\Delta m_N)u_N^2+{}^1/_4\Delta m_N u_{N+1}^2, & j=N,\end{cases}$$

then for $\varepsilon_T=\text{const}$ and $\varepsilon_V=0$ its variation during one step of time can be represented in the form

$$\frac{\overline{\Delta E}-\Delta E}{\Delta t}=F-F_{+},$$

where

$$F=\begin{cases} 0, & j=1,\\ \dfrac{u+\bar{u}}{4}\left[\bar{r}^{\nu}(P_{-}'+P')-\dfrac{1}{\bar{r}}(\bar{\eta}_{t-}\Sigma_{-}+\bar{\eta}_t\Sigma)\right], & j=2,3,\ldots,N,\\ \dfrac{u_{N+1}+\bar{u}_{N+1}}{2}\bar{r}_{N+1}^{\nu}P_b, & j=N+1\end{cases}$$

corresponds to the flux of mechanical energy through the left-hand and of the j-th interval. We recall that having restricted ourselves to the class of flows with a stationary centre $r=0$, we assume that the left boundary condition $u_1=\bar{u}_1=0$ is satisfied ($(u_1+\bar{u}_1)/\bar{r}_1=0$ in

formula (2.6)). Moreover, $P_{N+1}=P_b$ is the pressure on the right-hand boundary, and $\Delta m_{N+1}=\bar{\eta}_{s,N+1}=\bar{\eta}_{t,N+1}=0$.

In Eqs.(2.1)-(2.3) and (2.5) there are two independent coefficients of artificial viscosity $\bar{\eta}_s$ and $\bar{\eta}_t$. We assume that

$$\bar{\eta}_s=\begin{cases}0, & \Delta u\geqslant 0,\\ \dfrac{\rho\Delta m}{\langle\tilde{r}^{\nu}\rangle}(\mu_{s1}u_*-\mu_{s2}\Delta u), & \Delta u<0,\end{cases} \tag{2.8}$$

$$\bar{\eta}_t=\begin{cases}0, & \Delta u\geqslant 0,\\ \rho\Delta m\langle\tilde{r}^{\nu+2}\rangle(\mu_{t1}u_*-\mu_{t2}\Delta u), & \Delta u<0,\end{cases} \tag{2.9}$$

where u_* is the adiabatic speed of sound, and where the linear interpolation

$$\langle\tilde{r}^{\nu}\rangle=(1-\sigma_s)\tilde{r}^{\nu}+\sigma_s\tilde{r}_+^{\nu}, \tag{2.10}$$

$$\langle\tilde{r}^{\nu+2}\rangle=(1-\sigma_t)\tilde{r}^{\nu+2}+\sigma_t\tilde{r}_+^{\nu+2} \tag{2.11}$$

is used to centre the values of $\tilde{r}^{\nu}$ and $\tilde{r}^{\nu+2}$ at the middle of the interval. Each of the coefficients $\bar{\eta}_s$ and $\bar{\eta}_t$ is represented as a linear combination of a linear term with respect to the velocity of the gas (quadratic pseudoviscosity of the Neumann-Richtmyer type) and a term which is independent of the velocity of the gas (linear pseudoviscosity). The four non-dimensional coefficients μ_{s1}, μ_{s2}, μ_{t1}, and μ_{t2} offer a wide choice of the most appropriate version of the pseudoviscosity for a specific class of problems. If the Lagrange interval is expanded and $u_+\geqslant u$, then $\bar{\eta}_s$ and $\bar{\eta}_t$ both vanish. We remark that in the cylindrical and spherical cases the expansion defined in this way is not always accompanied by a decrease in the density ρ. The simpler non-divergent formulae (2.8) and (2.9) are preferable to the more complex divergent ones, since in practice they do not lead to any noticeable differences. Finally, scaling the coefficients $\bar{\eta}_s$ and $\bar{\eta}_t$ on Δm rather than on Δr seems to be more natural for the difference scheme (2.1)-(2.3), which relies on a uniform division with respect to the mass coordinate m. Moreover, such scaling produces better practical results.

Finally, the version of artificial viscosity proposed in the present paper contains six free non-dimensional parameters: the coefficient of linear scalar viscosity μ_{s1}, the coefficient of quadratic scalar viscosity μ_{s2}, the coefficient of linear t-viscosity μ_{t1}, the coefficient of quadratic t-viscosity μ_{t2}, and two free parameters $0<\sigma_s\leqslant 1$ and $0\leqslant\sigma_t\leqslant 1$ of the approximation.

3. *Numerical examples.*

The basic aim of the introduction of artificial viscosity is to describe properly the dissipation of kinetic energy on the fronts of shock waves. In the present section the results of the choice of coefficients for the pseudoviscosity introduced above are presented for two classes of shock wave flows: A) a shock wave reflected by a centre (an axis of a plane) of symmetry; B) a shock wave reflected by a centre with a subsequent stage of strong quasi-adiabatic contraction. In the case of an ideal gas with the equation of state

$$P=\rho T,\qquad \varepsilon=T/(\gamma-1)$$

test computations were carried out for the following analytic solutions.

The problem of a piston ($v=0$). The initial state is given as a plane layer of gas with $\gamma=5/3$ and with

$$0\leqslant r\leqslant R_0=1,\qquad \rho(0,r)=\rho_0=1,\qquad T(0,r)=0. \tag{3.1}$$

Starting from the instant $t=0$, the pressure $P_b(t)=P_1=3/4$ is applied at the right-hand boundary. Moreover, in the interval $0<t<t_1$ there is a strong shock wave moving towards the left-hand side with velocity

$$D_1=-\left(\frac{\gamma+1}{2}\frac{P_1}{\rho_0}\right)^{1/2}=-1.$$

After being reflected by the centre at the instant $t_1=1$, the shock wave moves towards the right-hand side with velocity

$$D_2=-(\gamma-1)u_1=-2\frac{\gamma-1}{\gamma+1}D_1=\frac{1}{2}.$$

The density and the pressure behind the front of the reflected shock wave are

$$\rho_2=\frac{\gamma}{\gamma-1}\rho_1=\frac{\gamma(\gamma+1)}{(\gamma-1)^2}\rho_0=10,\qquad P_2=\frac{3\gamma-1}{\gamma-1}P_1=\frac{9}{2}.$$

The values of the non-dimensional entropy

$$s=\frac{1}{\gamma-1}\ln T-\ln\rho$$

behind the fronts of the incoming and the reflected wave are $s_1=-3.89726$ and $s_2=-3.50035$.

The Guderley solution ($\nu=1, 2$). The initial state is a cylinder (a sphere) of ideal gas ($\gamma=5/3$) with parameters (3.1). Beginning from the instant $t=0$, the pressure of the piston $P_0(t)$ is applied on the external boundary. The pressure can be calculated in accordance with the selfsimilar Guderley solution /4, 6/ for the initial value $P_0(0)=P_0=5/3$. In this setting there is a shock wave of increasing strength moving towards the centre, which can be described by the Guderley solution for all t right up to the instant when it meets the piston (after being reflected from the centre). At the instant

$$t=t_1=\alpha R_0\left(\frac{2}{\gamma+1}\frac{\rho_0}{P_0}\right)^{1/\nu}=\alpha=\begin{cases}0.815625, & \nu=1,\\ 0.688377, & \nu=2\end{cases}$$

of arrival at the centre (α is the exponent of selfsimilarity) a state with constant density

$$\rho(t_1,r)=\rho_1=\begin{cases}7.018\rho_0, & \nu=1,\\ 9.550\rho_0, & \nu=2\end{cases}$$

is formed. The maximum compression behind the front of the reflected shock wave is

$$\max\{\rho(t>t_1,r)/\rho_0\}=\begin{cases}22.98, & \nu=1,\\ 32.28, & \nu=2.\end{cases}$$

Contraction and expansion of a sphere ($\nu=2$). One of the simplest selfsimilar solutions of the equations of ideal hydrodynamics is the adiabatic contraction and expansion of a sphere such that the density of the gas is independent of the position coordinate /7/:

$$\rho(t,r)=\rho(t)=\frac{3M}{4\pi R^3},\qquad u(t,r)=r\dot{R}/R,$$
$$P(t,r)=\frac{3MT_0}{4\pi R_0^3}\left(\frac{R_0}{R}\right)^{3\gamma}\left(1-\frac{r^2}{R^2}\right),$$

where the radius $R=R(t)$ can be determined from the equation

$$\ddot{R}=2T_0R_0^{3\gamma-3}R^{2-3\gamma}.$$

If the physical mass of the sphere is M, then the solution is completely defined by the central temperature T_0 and the radius R_0, given at the instant of maximum contraction. Tensor calculations lead to the values

$$\gamma=5/3,\quad M=4\pi/3,\quad R_0=T_0=1.$$

In the problem of expansion, the initial state, the instant of maximum contraction, and the final state are characterized by the parameters $t=t_1=4$, $R=R_1=5.744563$, and $\dot{R}=\dot{R}_1=1.392621$. For the problem of contraction the initial and final states are exchanged and the sign of the velocity is changed accordingly.

Table 1

Parameter	Sphere				Cylinder	
	1	2	3	4	5	6
μ_{x1}	0	0.1	0.1	0.1	0.1	0.1
μ_{x2}	2	0.25	0.25	0	0.25	0
σ_x	1	1	1	1	1	1
μ_{t1}	0	0	0.2	0	0	0
μ_{t2}	0	0	0	2	0	2
σ_t	0	0	0	0.1	0	0.1
ρ_1	4.44	9.96	10.1	12.7	8.13	10.2
	8.73	10.8	10.7	12.9	7.19	9.58
	9.76	8.93	8.84	10.4	6.03	8.30
	10.1	8.33	8.28	10.2	6.37	7.61
Δx_2	1.12	−0.13	−0.15	−0.72	0.23	−0.30
	0.88	0.19	0.23	−0.39	0.07	−0.64
	0.60	0.12	0.17	−0.23	0.02	−0.39
	0.47	0.09	0.14	−0.16	0	−0.27

In the first column of Table 1 (the Guderley solution) an example of a test computation for a converging spherical shock wave with the Neumann-Richtmyer quadratic scalar pseudo-viscosity is presented. To facilitate comparison with the exact solution, the values of the density ρ_1 at the instant of cumulation are chosen in the first four mass intervals (counting

from the centre) along with the surplus entropy Δs_2 (compared with the exact solution) in the same intervals behind the reflected wave. This and all subsequent computations were carried out for a uniform division with respect to the mass into 100 intervals (into 40 intervals for the problem of a piston). Since there is a limit imposed on the steps in time by the Courant condition (with a margin coefficient $K_h=0.5$) and by the pseudoviscosity /1/, it takes on the average ~10 steps of time for the front to pass through one mass interval. As one can see from the first column of Table 1, the classical Neumann-Richtmyer pseudoviscosity leads to a strong superfluous warming of the centre. We remark that the maximum value we set for the parameter $\sigma_i=1$ results in minimum warming of the central (the first) interval (see (2.8) and (2.10)).

The combination of the linear and the quadratic scalar viscosity (column 2 of Table 1) turned out to be the best from the point of view of the description of flows of class A (reflection of a shock wave from the centre without subsequent adiabatic contraction). However, the coefficients μ_{e1} and μ_{e2} found in this way are relatively small, which results in relatively large ±30% oscillations of the pressure behind the front (see Fig.1). The oscillations of the pressure can be suppressed by adding the linear t-viscosity (up to ±13% for $\mu_{t1}=0.2$), which has a negligible effect on the warming of the centre (see column 3 of Table 1 and Fig.1). Attempts to describe properly the reflection of a shock wave from the centre using one t-viscosity only yield either insufficient or substantially non-uniform warming of the central intervals. It is obviously also possible to find a suitable combination of the quadratic scalar viscosity and the t-viscosity.

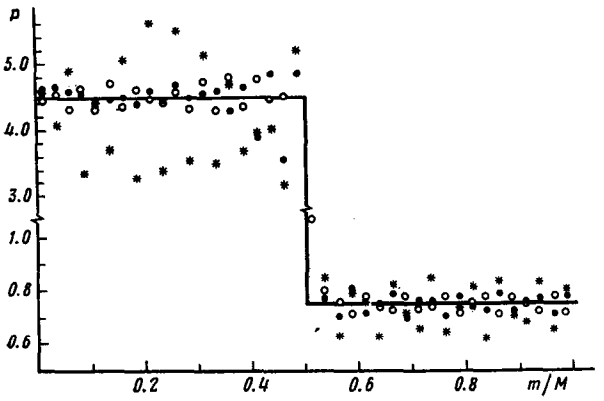

Fig.1

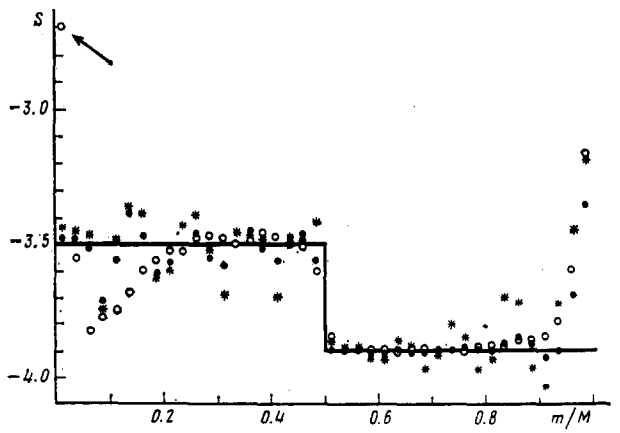

Fig.2

The coefficients of viscosity given in columns 2 and 3 of Table 1 also produce good results in the cylindrical and plane cases (see column 5 of Table 1 and Figs.1 and 2).

For problems of class B, the combinations $\mu_{e1}=0$, $\mu_{e2}=2$ and $\mu_{e1}=0.1$, $\mu_{e2}=0.25$ are unsuitable since they result in significant parasitic generation of entropy in the central intervals at the stage of adiabatic contraction (see columns 1 and 2 of Table 2). In this case the superposition of the linear scalar viscosity and the quadratic t-viscosity with $\mu_{e1}=0.1$, $\mu_{t2}=2$, and with the approximation coefficient $\sigma_i=0.1$ turns out to be the best one. For this superposition the deficit of entropy arising as the shock wave is being reflected by the centre (see column 4 of Table 1) is almost completely compensated by an increase in entropy during the stage of adiabatic contraction (see column 4 of Table 2). We remark that for the problem of the contraction of a sphere, large perturbations of the density and

entropy accumulate in a few intervals adjacent to the external boundary: for the computation corresponding to column 4 of Table 2, $\rho_1=0.46$ (rather than $\rho_1=1$) in the last interval with $j=100$. However, as a whole, the description of the contraction of a sphere is completely satisfactory (compare the value of R_1 in Table 2 with the exact value $R_1=1$).

Table 2

Parameter		1	2	3	4
μ_{s1}		0	0.1	0.1	0.1
μ_{s2}		2	0.25	0.25	0
σ_s		1	1	1	1
μ_{t1}		0	0	0.2	0
μ_{t2}		0	0	0	2
σ_t		0	0	0	0.1
ρ_1	{	0.37	0.66	0.65	0.68
		0.66	0.83	0.82	0.88
		0.75	0.87	0.87	0.91
		0.80	0.89	0.90	0.92
Δs_1	{	2.37	0.96	0.96	0.87
		0.93	0.38	0.39	0.25
		0.61	0.26	0.27	0.18
		0.43	0.20	0.21	0.14
R_1		1.144	1.052	1.049	1.035

For the problem of the expansion of a sphere, the artificial viscosity vanishes in accordance with (2.8) and (2.9). However, until the instant $t=4$ there is a small deficit of entropy $\Delta s_1=-0.01--0.04$ due to the errors of the finite-difference approximation. The deficit decreases linearly as the margin coefficient K_h decreases in accordance with the Courant condition. For $K_h=0.5$, small-scale vibrations with amplitude $\Delta\rho/\rho\approx\pm5\%$ are being excited at the background of the uniform expansion. Obviously, the vibrations result from the fact that the half-sums of the corresponding velocities of the new and old time layers are used in (2.6) and (2.7) instead of the velocities $\bar{u}$, $\bar{u}_+$ of the new layer (this is necessary in order that the system should be conservative with respect to the energy for $\varepsilon_V=0$ and $\varepsilon_T=\text{const}$). For $K_h=0.3$ there is no excitation of vibrations.

In Figs.1 and 2 the results of the solution of the problem of a piston at the instant $t_2=1.1$ are presented for the first three combinations of the coefficients of viscosity from Table 1. Circles ○ are used here for $\mu_{s1}=\mu_{t1}=\mu_{t2}=0$, $\mu_{s2}=2$, and $\sigma_s=1$, asterisks are used for $\mu_{s1}=0.1$, $\mu_{s2}=0.25$, $\sigma_s=1$ and $\mu_{t1}=\mu_{t2}=0$, and filled circles ● are used for $\mu_{s1}=0.1$, $\mu_{s2}=0.25$, $\sigma_s=1$, $\mu_{t1}=0.2$ and $\mu_{t2}=\sigma_t=0$. One can see that the combination of $\mu_{s1}=0.1$ and $\mu_{s2}=0.25$ makes it possible to eliminate almost completely the effect of warming of the wall. Moreover, the oscillations behind the front of the shock wave can be significantly suppressed by adding the t-viscosity.

Summing up, we conclude from the above calculations that the complex form of artificial viscosity proposed in this paper enables one to reduce significantly some undesirable side-effects characteristic of the Neumann-Richtmyer pseudoviscosity by choosing the values of the six free parameters for the class of one-dimensional flows of interest.

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